NONREGULAR ULTRAFILTERS ON $\omega_2$

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Abstract. We obtain lower bounds for the consistency strength of fully nonregular ultrafilters on $\omega_2$.

1. Introduction

A nonregular ultrafilter is a notion which arose from classic questions in model theory about cardinalities of ultrapowers (see Chang and Keisler [1]). Nonregularity is a weakening of countable completeness, and allows for the possibility of a small ultrapower: if an ultrafilter $U$ on $\omega_1$ has the property that $|\omega_1 \omega/U| = \omega_1$, then $U$ must be nonregular.

Although in $ZFC$ there is never a countably complete ultrafilter over a cardinal like $\omega_n$, it is consistent relative to large cardinals that there is a nonregular ultrafilter on $\omega_n$ ($n \geq 1$). An upper bound for the consistency strength of the existence of a nonregular ultrafilter on $\omega_1$ is $\omega$ many Woodin cardinals (see Woodin [13]). Laver [12] showed that if there is an $\omega_1$-dense uniform ideal on $\omega_1$ and ♦ holds, then there is a nonregular ultrafilter on $\omega_1$. Huberich [9] removed the assumption of ♦.

The known consistency upper bound for a fully nonregular ultrafilter on $\omega_2$ is higher. Foreman, Magidor, and Shelah [8] obtained such an ultrafilter from an almost huge cardinal. Foreman [7] obtains such an ultrafilter $U$ from a huge cardinal, with many additional properties including the small ultrapower property which appears in part 2 of Theorem 1 below.$^1$

Ketonen [11] proved that if there is a nonregular ultrafilter on $\omega_1$, then $\text{0}^\sharp$ exists. This was later improved by Donder, Jensen, and Koppelberg [5], and then by Deiser and Donder [6]. Deiser and Donder

$^1$Foreman’s model actually satisfied $|\omega_2 \omega/U| = \omega_1$, which is stronger than the condition appearing in the hypothesis of Theorem 1.
showed that the consistency strength is at least a stationary limit of measurables. In fact, their proof can be slightly modified to show: if there is a nonregular ultrafilter on \( \omega_1 \) then either 0-sword exists or else
\[
\{ \nu < \omega_2 | \text{cf}(\nu) = \omega_1 \text{ and } \nu \text{ is measurable in } K \}
\]
contains an \( \omega_1 \)-club in \( V \).\(^2\) In either case, there is an inner model \( M \) such that \( V \) sees an \( \omega_1 \)-club through \( M \)'s measurables below \( \omega_2 \).

We build on their argument to obtain lower consistency bounds for fully nonregular ultrafilters on \( \omega_2 \):

**Theorem 1.** Suppose there is a fully nonregular ultrafilter \( U \) on \( \omega_2 \). Then:

1. There is an inner model with a cardinal \( \kappa \) of Mitchell order \( \kappa^+ \). In particular, if 0-pistol does not exist, then letting \( \kappa = \omega_3^V \), the Mitchell order of \( \kappa \) in the core model \( K \) is at least \( \kappa^+ \).
2. If \( \omega_1/U = \omega_2 \), then there are mice extenders with multiple generators (the consistency strength of this is a bit higher than a measurable cardinal \( \kappa \) with Mitchell order \( \kappa^{++} \)).

This paper is organized as follows. Section 2 reviews filtrations and canonical functions. Section 3 reviews some facts about nonregular ultrafilters and the related notion of a weakly normal ultrafilter. Section 4 defines the “bounding construction” and proves essential facts about the construction, most of which are abstracted from [6]. Section 5 is the proof of part (1) of Theorem 1, and Section 6 is the proof of part (2) of Theorem 1.

2. Filtrations and Canonical Functions

The notation \( S^\kappa_\lambda \) will denote the set \( \{ \alpha < \kappa | \text{cf}(\alpha) = \lambda \} \). We will also write \( S^m_n \) instead of \( S^\omega_\omega^m \).

As in Deiser and Donder [6] we will make extensive use of canonical functions. If \( A \) is a set of cardinality \( \kappa \), a filtration of \( A \) is a continuous \( \subset \)-increasing sequence \( \langle A_\alpha | \alpha < \kappa \rangle \) such that \( |A_\alpha| < \kappa \) for every \( \alpha < \kappa \) and \( A = \bigcup_{\alpha < \kappa} A_\alpha \). If \( \kappa \) is regular and uncountable then any two filtrations of \( A \) agree on a club. So if \( \nu < \kappa^+ \) and \( \langle A_\alpha^\nu | \alpha < \kappa \rangle, \langle B_\alpha^\nu | \alpha < \kappa \rangle \) are filtrations of \( \nu \), then the functions \( \alpha \mapsto \text{otp}(A_\alpha^\nu) \) and \( \alpha \mapsto \text{otp}(B_\alpha^\nu) \) agree on a club. The equivalence class\(^3\) of \( \alpha \mapsto \text{otp}(A_\alpha^\nu) \) is called the

\(^2\)0-sword is an object analogous to 0-sharp; it is an iterable mouse with a top measure of Mitchell order 1. In the absence of 0-sword, the core model \( K \) can be constructed and has many properties that \( L \) has in the absence of 0-sharp. In this paper we will use 0-pistol instead, which is a stronger object than 0-sword; see Section 4.

\(^3\)modulo the equivalence relation on \( ^*V \) defined by agreement on a club.
$\nu$-th canonical function on $\kappa$. There are also inductive characterizations of the canonical functions. Let $h_0$ be the zero function (on $\kappa$); if $h_\nu$ is defined let $h_{\nu+1} := h_\nu + 1$; and for limit $\nu < \kappa^+$ fix any cofinal increasing\(^4\) sequence $\langle \nu_\alpha | \delta < \text{cf}(\nu) \rangle$ in $\nu$ and define:

\[
(1) \quad h_\nu(\alpha) = \begin{cases} 
\sup_{\delta < \text{cf}(\nu)} h_{\nu_\beta}(\alpha) & \text{if \text{cf}(\nu) < \kappa} \\
\sup_{\beta < \alpha} \nu_\beta(\alpha) & \text{if \text{cf}(\nu) = \kappa}
\end{cases}
\]

Then for each $\nu < \kappa^+$, $h_\nu$ represents the $\nu$-th canonical function on $\kappa$.

We will often use constructions of the following form, where $\kappa$ is regular and uncountable. Fix some large regular $\theta \geq \kappa^+$ and a wellordering $\Delta$ of $H_\theta$. For each $b \in H_\theta$ of cardinality $\kappa$ let $\text{Filt}_b = \langle \text{Filt}_b(\alpha) | \alpha < \kappa \rangle$ denote the $\Delta$-least filtration of $b$; note this can just be viewed as $\langle f^b \alpha | \alpha < \kappa \rangle$ where $f^b$ is some surjection $\kappa \to b$. Let $S^b$ be the collection of $X \in P_\kappa(H_\theta)$ such that $X \times (H_\theta, \in, \Delta, \{b\})$ and $X \cap \kappa$ is transitive; for each such $X$ let $\alpha_X := X \cap \kappa$. Let $\bar{S}^b := \{\alpha_X | X \in S^b\}$.

For each $X \in S^b$ let $\sigma_X : H_X \to H_\theta$ be the inverse of the Mostowski collapsing map for $X$. Since $b \in X < (H_\theta, \Delta, \in)$ then $\text{Filt}_b \in X$ and it is straightforward to show that $X \cap b = \text{Filt}_b X$.

Now suppose $|\text{trcl}(b)| < \kappa^+$ (note since we’re assuming $|b| = \kappa$, then this means $|\text{trcl}(b)| = \kappa$). If $X, Y$ are both in $S^b$ and $\alpha_X = \alpha_Y =: \alpha$, then $\sigma_X^{-1}(b) = \sigma_Y^{-1}(b)$; this is because $X \cap \text{trcl}(b) = \text{Filt}_b \text{trcl}(b) = Y \cap \text{trcl}(b)$. Thus the following definition does not depend on the particular choice of $X$:

**Definition 2.** If $b \in H_{\kappa^+}$ and $\alpha \in \bar{S}^b$, $b_\alpha$ is defined to be $\sigma_X^{-1}(b)$ where $X$ is any element of $S^b$ with $\alpha_X = \alpha$. Sometimes we will refer to the function $\alpha \mapsto b_\alpha$ as the canonical function indexed by $b$ and denote it $h_b$.

Finally, suppose $\nu \in S^\kappa_{\kappa^+}$, $\langle \nu_\alpha | \delta < \kappa \rangle$ is a cofinal sequence in $\nu$, and $X \in S^\nu$. Then $X \cap \{\nu_\alpha | \delta < \kappa \} = \{\nu_\alpha | \delta < \alpha_X \}$ and:

\[
(2) \quad \begin{align*}
\sigma_X^{-1}(\nu) &= h_\nu(\alpha_X) \\
\sigma_X^{-1}(\langle \nu_\alpha | \delta < \kappa \rangle) &= \langle h_{\nu_\alpha}(\alpha_X) | \delta < \alpha_X \rangle
\end{align*}
\]

3. **Weakly normal and nonregular ultrafilters**

Let $U$ be an ultrafilter on a regular uncountable cardinal $\kappa$. Consider the class $\mathcal{G}$ of partial functions $f : \kappa \to V$ such that $\text{dom}(f) \in U$. For functions $f, g \in \mathcal{G}$, $f =_U g$ means that $\{\alpha \in \text{dom}(f) \cap \text{dom}(g) | f(\alpha) = g(\alpha) \} \in U$ (i.e. $\text{ult}(V, U) = \langle [f]_U = [g]_U \rangle$). The notations $f \in_U g$ and $f <_U g$ (if $f, g$ are ordinal-valued) are defined similarly.

\(^4\)Unless stated otherwise, whenever we specify a sequence $\langle \nu_\delta | \delta < \text{cf}(\nu) \rangle$, we will always require that it is increasing.
Definition 3. Let $\kappa$ be a regular uncountable cardinal, and $U$ a uniform ultrafilter over $\kappa$. Let $\lambda < \kappa$ be a cardinal. $U$ is $(\lambda, \kappa)$-nonregular iff whenever $A \subset U$ and $|A| = \kappa$ then there is a $A' \subset A$ of cardinality $\lambda$ such that $\cap A' \neq \emptyset$.

If $\kappa = \lambda^+$, then $\lambda$ is clearly the largest possible cardinal where $U$ might be $(\lambda, \kappa)$-nonregular. If $U$ is $(\lambda, \lambda^+)$-nonregular then we say it is fully nonregular.

Just as nonregularity is a weakened version of $\kappa$-completeness of an ultrafilter, the following is a weakened notion of a normal ultrafilter:

Definition 4. A uniform ultrafilter $U$ over $\kappa$ is weakly normal iff whenever $f : \kappa \to \kappa$ is a regressive (partial) function with $\text{dom}(f) \in U$, then there is a $\delta < \kappa$ and $A \in U$ such that $f \upharpoonright A$ is bounded by $\delta$.

If $\lambda$ is regular then the existence of a fully nonregular ultrafilter on $\lambda^+$ is equivalent to the existence of a weakly normal ultrafilter on $\lambda^+$ which concentrates on $S_{\lambda^+}^\lambda$. In fact, every weakly normal ultrafilter concentrating on $S_{\lambda^+}^\lambda$ is fully nonregular, and any fully nonregular ultrafilter projects to a weakly normal ultrafilter in the Rudin-Keisler order, via a “least unbounded function” modulo the nonregular ultrafilter; see Kanamori [10]. Also, if $U$ is uniform and weakly normal, it is easy to see that $U$ extends the club filter.

Lemma 5. (Diagonalization Lemma). Suppose $U$ is a weakly normal ultrafilter and $X_\xi \in U$ for every $\xi < \kappa$. Then $\{\alpha < \kappa | \alpha \in X_\xi \text{ for cofinally many } \xi < \alpha \} \in U$.

Note then if $f_\xi$ is a function with domain $X_\xi \in U$ for each $\xi < \kappa$, then it’s natural to define $\text{diagsup}_{\xi < \kappa} f_\xi$ as the function with domain $\{\alpha < \kappa | \alpha \in \text{domain}(f_\xi) \text{ for cofinally many } \xi < \alpha \}(\in U)$ which sends $\alpha \mapsto \sup_{\xi < \alpha} f_\xi(\alpha)$. It’s easy to see that this diagsup is $\geq_U f_\xi$ for every $\xi < \kappa$.

For the remainder of the section, assume $U$ is a weakly normal ultrafilter on $\omega_2$ and $S_2^\alpha \in U$.

The proof of Lemma 5.3 in [6] goes through to show:

Lemma 6. There is no collection $\mathcal{F}$ of functions such that:

- $|\mathcal{F}| = \omega_3$
- $\text{dom}(f) \in U$ and $\text{range}(f) \subset \omega_1$ for every $f \in \mathcal{F}$
- $\{\xi \in \text{dom}(f) \cap \text{dom}(g) | f(\xi) = g(\xi)\}$ is nonstationary for every distinct pair $f, g$ in $\mathcal{F}$.

The collection $\mathcal{F}$ in the statement of Lemma 6 is often called a “transversal sequence.”
Corollary 7. There is no pair \( h, G \) such that:
- \( G \) is an \( \omega_3 \)-sized collection of partial functions from \( \omega_2 \to \omega_2 \) whose domains are in \( U \);
- For every distinct \( f, g \in G \): there are only nonstationarily many \( \alpha \in \text{dom}(f) \cap \text{dom}(g) \) such that \( f(\alpha) = g(\alpha) \);
- \( h : \omega_2 \to \omega_2 \) and for every \( f \in G \), \( f <_U h \).

Proof. For each \( \beta < \omega_2 \) fix an injective \( \psi_\beta : \beta \to \omega_1 \). For each \( f \in G \), if \( \alpha \in \text{dom}(f) \) and \( f(\alpha) < h(\alpha) \), let \( f'(\alpha) := \psi_{h(\alpha)}(f(\alpha)) \). Then \( \{ f' \mid f \in G \} \) is an \( \omega_3 \)-sized collection of functions which has the properties listed in Lemma 6, a contradiction. \( \square \)

The proof of Corollary 5.4 in [6] shows:

Lemma 8. For every \( f : \omega_2 \to \omega_2 \), there is a \( \nu < \omega_3 \) such that \( f <_U h_\nu \).

Finally, the proofs of Lemma 5.5 and Corollary 5.6 in [6] show:

Lemma 9. If \( \text{cf}(\nu) = \omega_2 \) then \( [h_\nu]_U \) is the least upper bound of \( \{ [h_\tau]_U \mid \tau < \nu \} \) in \( \text{ult}(V,U) \).

4. The Bounding Construction

Throughout the rest of the paper, \( K \) denotes the core model for non-overlapping extenders, built under the assumption that 0-pistol does not exist. Basic facts about 0-pistol and \( K \) can be found in Chapter 8 of [14]. In particular, \( K \) is capable of having a strong cardinal, but comparisons of mice are still linear. Note that if 0-pistol exists, then there is a sharp for an inner model with a strong cardinal, so the conclusion of Theorem 1 would hold and we’d be finished.

Let \( E \) be \( K \)’s extender sequence, and for each \( X \in P_{\omega_2}(H_{\omega_3}) \) such that \( X < (H_{\omega_3}, \in, E \mid \omega_3, ...) \) let \( K_X := \sigma_X^{-1}[K \cap X] \), where \( \sigma_X \) is the inverse of the Mostowski Collapse of \( X \).

Let \( S \) be the collection of \( X \in P_{\omega_2}(H_{\omega_3}) \) such that \( X < (H_{\omega_3}, \in, \Delta, ...) \), \( \alpha_X := X \cap \omega_2 \in S_1^2 \), and \( \lambda_X := \sup(X \cap \omega_3) \in S_1^3 \) (these are sometimes called sets of uniform cofinality \( \omega_1 \)). Then:

\[ X \cap \omega_3 \text{ is an } \omega \text{-closed set of ordinals for every } X \in S. \] (so \( \sigma_X : H_X \to H_{\omega_3} \) is continuous on \( \text{cof}(\omega) \).) To see that (3) holds, suppose \( \nu \in S_0^3 \cap \text{Lim}(X) \). Since \( \lambda_X \) has uncountable cofinality, there is some \( \nu' \in X \cap \omega_3 \) such that \( \nu < \nu' \). Then there is a bijection \( g : \omega_2 \to \nu' \) such that \( g \in X \); since \( \alpha_X \) has uncountable cofinality, there is a \( \beta < \alpha_X \) such that \( (g'' \beta) \cap \nu \) is cofinal in \( \nu \). But \( g'' \beta \subset X \), so \( X \cap \nu \) is cofinal in \( \nu \).
For \( \nu \in \omega_3 \) consider the set \( S \cap S^\nu \). Let \( \bar{S}^\nu \) be the projection of \( S \cap S^\nu \), i.e. \( \bar{S}^\nu := \{ \alpha_X | X \in S \cap S^\nu \} \). Note that \( \bar{S}^\nu \) contains an unbounded subset which is closed under limits of cofinality \( \omega_1 \). Since \( U \) extends the club filter and \( S^2 \subseteq U \) (by assumption), then \( \bar{S}^\nu \subseteq U \). For \( X \in S^\nu \):

\[
(4) \quad \sigma_X^{\nu} \text{ denotes the map } \sigma_X \upharpoonright h_\nu(\alpha_X). 
\]

Note that for any \( X \) which has the sequence \( \langle \nu_\xi \delta < cf(\nu) \rangle \) as an element,\(^6\) it holds that \( h_\nu(\alpha_X) = otp(X \cap \nu) = \sigma_X^{-1}(\nu) \).

If \( X, Y \) are both in \( S^\nu \) and \( \alpha_X = \alpha_Y =: \alpha \), then \( X \cap \nu = Y \cap \nu \) (they both equal \( \text{Filt}^\nu_X \)). So \( \sigma_X^{\nu} = \sigma_Y^{\nu} \). In other words, the map \( \sigma_X^{\nu} \) is independent of our choice of \( X \in S^\nu \) with \( \alpha_X = \alpha \). So the following definition makes sense:

**Definition 10.** If \( \nu \in \omega_3 \) and \( \alpha \in \bar{S}^\nu \), we define \( \sigma_\alpha^{\nu} \) as the map \( \sigma_X^{\nu} \), where \( X \) is any element of \( S^\nu \) with \( \alpha_X = \alpha \).

Now for any \( X \in S \), consider the coiteration of \( K_X \) with \( K \). Let \( \Omega_X \) denote the length of the \( K_X \) vs. \( K \) coiteration, and

\[
\langle N_i^X, \pi_i^X, \mu_i^X, \kappa_i^X, \tau_i^X, \delta_i^X, (N_i^X)^* | i \leq j \leq \Omega_X \rangle
\]

denote the objects on the \( K \) side of the coiteration; i.e. \( N_i^X \) is the \( i \)-th iterate, \( \pi_i^X \) the iteration map, \( \mu_i^X \) the iteration index, \( \kappa_i^X \) the critical point, \( \tau_i^X \) the smaller of the cardinal successors of \( \kappa_i^X \) in the \( i \)-th iterates, \( \delta_i^X \) the maximal segment of \( N_i^X \) where \( \tau_i^X \) is a cardinal, and \( (N_i^X)^* = N_i^X \upharpoonright \delta_i^X \).

Since each \( X \in S \) has an \( \omega \)-closed intersection with \( \omega_3 \) (see (3)), Lemma 39 from [2] applies,\(^7\) so \( \alpha_i^{+K_X} \) is not a cardinal in \( K \), the \( K \) side of the coiteration truncates to an \( \omega_1 \)-sized mouse either at stage 0 or at stage 1, and the \( K_X \) side of the coiteration is trivial. Let \( i_0^X \in \{ 0, 1 \} \) denote the first truncation stage. So \( |(N_{i_0^X})^*| < \omega_2 \). Since \( |(N_{i_0^X})^*| < \omega_2 \) and \( |K_X| < \omega_2 \), then \( |(N_i^X)^*| < \omega_2 \) for every \( i \in [i_0^X, \Omega_X) \), and \( \Omega_X < \omega_2 \). Also, since the \( K_X \) side is trivial in the coiteration, we have:

\[
(5) \quad \mu_i^X = \sigma_{K_X}(\kappa_i^X)
\]

for every stage \( i \) of the coiteration.

For \( \nu \in \omega_3 \) and \( X \in S^\nu \) let \( K_X^\nu = K_X|\sigma_X^{-1}(\nu) \). Let \( \theta_X^{\nu} \) denote the least ordinal \( \iota \) such that \( \mu_i^X \) is either not defined or is at least

\(^5\)This is standard; construct an elementary \( \in \)-chain \( (X_\beta | \beta < \omega_2) \) consisting of models from \( S \cap S^\nu \), such that for \( \beta \) of cofinality \( \omega_1 \), \( X_\beta = \bigcup_{\beta < \beta} X_\beta \). Then the projection of this chain contains an \( \omega_1 \)-club in \( \omega_2 \).

\(^6\)where this \( \vec{\nu} \) was the cofinal sequence in \( \nu \) used to define the “official” representative \( h_\nu \) for the \( \nu \)-th canonical function in (1).

\(^7\)This was an argument due to Mitchell.
height$(K^\nu_X) = h_\nu(\alpha_X)$. In other words: $\theta^\nu_X$ is equal to the length of the $K$ vs. $K^\nu_X$ coiteration, unless the height of $K^\nu_X$ (which is $h_\nu(\alpha_X)$) indexes an extender in $N^X_\theta$; in this latter case the length of the $K$ vs. $K^\nu_X$ coiteration will equal $\theta^\nu_X + 1$. Define $N^X_\theta := N^\nu_{\theta^\nu_X}$. Similarly to the discussion before Definition 10, if $\alpha \in \bar{S}^\nu$ then the object $K^\nu_X$ does not depend on the particular choice of $X \in S^\nu$ with $\alpha_X = \alpha$. So the following definition makes sense:

**Definition 11.** If $\nu \in \omega_3$ and $\alpha \in \bar{S}^\nu$: define $K^\nu_{\alpha}, N^\nu_{\alpha}, \theta^\nu_{\alpha}$ as $K_X^\nu, N_X^\nu, \theta_X^\nu$ (respectively) where $X$ is any element of $S^\nu$ with $\alpha = \alpha_X$.

Given any $\nu < \nu'$ in $\omega_3$, $K^\nu_{\alpha}$ is an initial segment of $K^\nu_{\alpha'}$ for all but nonstationarily many $\alpha \in S^\nu_2$, and neither move in their coiteration with $K$. Then the coiteration of $K$ with $K^\nu_{\alpha}$ is an initial segment of the coiteration of $K$ with $K^\nu_{\alpha'}$. This motivates the next definition:

**Definition 12.** Let $\nu < \nu'$ and $\alpha \in \bar{S}^\nu \cap \bar{S}^\nu'$. Let $M$ be any mouse such that $K^\nu_{\alpha}$ is an initial segment of $M$, and let $\pi_{i,j}$ denote the iteration maps on the $K$ side of the $K$ vs. $M$ coiteration. $\pi^\nu_{\alpha,\alpha'}$ will denote the (possibly partial) iteration map $\pi_{\theta^\nu_{\alpha'},\theta^\nu_{\alpha}}$.

Of course, a priori it could happen that $\theta^\nu_{\alpha'} = \theta^\nu_{\alpha}$; we will see this is not usually the case.

Suppose $\nu \in S^\nu_2$ and $t_\nu = \langle \nu_\delta|\delta < \omega_2 \rangle$ is a (not necessarily continuous) sequence cofinal in $\nu$. Pick any $X \in S^\nu$. Then $X \cap \{ \nu_\delta|\delta < \omega_2 \} = \{ \nu_\delta|\delta < \alpha_X \}$; so $X \in S^\nu_s$ for every $\delta < \alpha_X$, and $X \cap \nu_\delta = \text{Filt}^\nu_{\alpha_X}$. Then

- For every $\delta < \alpha_X$: $K^\nu_{\alpha_X} = \sigma_X^{-1}(K|\nu)$ is an initial segment of $K^\nu_{\alpha_X} = \sigma_X^{-1}(K|\nu)$ and the height of $K^\nu_{\alpha_X}$ is $\text{otp}(\text{Filt}^\nu_{\alpha_X}) = h_\nu(\alpha_X)$;
- The height of $K^\nu_{\alpha_X}$ is $\text{otp}(\text{Filt}^\nu_{\alpha_X}) = h_\nu(\alpha_X)$ and $K^\nu_{\alpha_X} = \bigcup_{\delta < \alpha_X} K^\nu_{\alpha_X}$.

Since the $K_X$ side of the coiteration is trivial and $K^\nu_X = \bigcup_{\delta < \alpha_X} K^\nu_{\alpha_X}$, then $\theta^\nu_X = \sup_{\delta < \alpha_X} \theta^\nu_{\alpha_X}$. This follows from the way we defined the ordinal $\theta^\nu_X$; it’s possible that the length of the coiteration of $K$ with $K^\nu_X$ is actually $\theta^\nu_X + 1$, which may happen if $h_\nu(\alpha_X) = \text{ht}(K^\nu_X)$ is a coiteration index. However, we will see later that this does not happen if $K$ has no overlapping extenders.

Let $\nu < \omega_3$. Recall for every $\alpha \in \bar{S}^\nu$, $\alpha + K^\nu_X$ is not a cardinal in $K$ and there is a truncation by stage 1 on the $K$ side. In particular, $(N^\nu_{\alpha})^*$ has cardinality $< \omega_2$. Let $Q^\nu_{\alpha}$ denote the collection of all mice which are iterates of $(N^\nu_{\alpha})^*$ where the iteration is of length $\leq 2$; i.e. the collection of all possible iterates of $(N^\nu_{\alpha})^*$ for at most 2 stages past stage $\theta^\nu_{\alpha}$ . Let $q^\nu_{\alpha} := \sup\{ \text{height}(N)|N \in Q^\nu_{\alpha}\}$; note $q^\nu_{\alpha} < \omega_2$. Define

\[
\psi_\nu : \bar{S}^\nu \to \omega_2 \text{ by } \alpha \mapsto q^\nu_{\alpha}
\]
The ordinals in $S_K$ and we have not yet made any assumptions about the $\alpha$-strength of $\nu$; see Section 4. There is a key difference in the construction here and that in [4]: in the current construction, we have not yet made any assumptions about the $K$-strength of the ordinals in $S_\omega^2 \cap K$, as this is not needed for the bounding construction. Recursively define a closed unbounded subset $D$ of $\omega_3$ as follows: suppose some initial segment $D$ of $D$ has been defined. If $D$ has no largest element, we choose $\sup(\bar{\nu})$ to be the next element of $D$. Otherwise $D$ has a largest element, say $\nu$. Then the next element of $D$—which we will denote $\nu^*$—is chosen so that $\psi_\nu < h_{\nu^*}$; this is possible by Lemma 8.

Then for every $\nu \in D$ there are $U$-many $\alpha$ such that $\psi_\nu(\alpha) < h_{\nu^*}(\alpha)$. This is the scenario described before (7), so

$$\theta^{\nu^*}_\nu < U \theta^{\nu^*_\nu}_\nu \quad \text{(in fact } \theta^{\nu^*_\nu}_\nu + 1 < U \theta^{\nu^*_\nu}_\nu)$$

Here, the symbol $\theta^{\nu^*_\nu}_\nu$ denotes the (partial) function $\alpha \mapsto \theta^{\nu^*_\nu}_\nu$. We will also use the notation $\{\theta^{\nu^*_\nu}_\nu < \theta^{\nu^*_\nu}_\nu\}$ to denote the set of $\alpha$ such that both $\theta^{\nu^*_\nu}_\nu$ and $\theta^{\nu^*_\nu}_\nu$ are defined and $\theta^{\nu^*_\nu}_\nu < \theta^{\nu^*_\nu}_\nu$ (and similarly for other functions on $\omega_2$).

Then for $\alpha \in \{\theta^{\nu^*_\nu}_\nu < \theta^{\nu^*_\nu}_\nu\}$ let $\kappa^{\nu^*_\nu}_\alpha := \kappa^{\nu^*_\nu}_\alpha$ where $X \in S^K \cap S^{\nu^*_\nu}$ and $\alpha = \alpha_X$—i.e. the critical point at stage $\theta^{\nu^*_\nu}_\nu$ of the $K$ versus $K^{\nu^*_\nu}$ coiteration. Note that if $\nu^*$ is some element of $D$ which is at least $\nu^*$ and $\alpha = \alpha_X$ for some $X \in S^{\nu^*_\nu} \cap S^{\nu^*_\nu}$ then $\kappa^{\nu^*_\nu}_\alpha$ is also the critical point at stage $\theta^{\nu^*_\nu}_\nu$ of the $K$ versus $K^{\nu^*_\nu}$ coiteration. Similarly define $\mu^{\nu^*_\nu}_\alpha$ as the iteration index at stage $\theta^{\nu^*_\nu}_\alpha$ of the $K$ versus $K^{\nu^*_\nu}$ coiteration. Note that there may be a truncation at stage $\theta^{\nu^*_\nu}_\nu$. We will arrange that, often enough, such truncations do not occur; this is not essential to the argument but it will provide some simplifications.

The construction of $D$ guarantees that for every $\nu \in D$:

$$\psi_\nu < U h_{\nu^*} \leq U \psi_{\nu^*}$$

and

$$h_{\nu^*} \leq U \psi_{\nu^*} \leq U \kappa^{\nu^*}_{\theta^{\nu^*_\nu}_\nu + 1} \leq U \kappa^{\nu^*}_{\theta^{\nu^*_\nu}_\nu} \leq U \psi_{\nu^*}$$

by def. $\kappa^{\nu^*}_{\theta^{\nu^*_\nu}_\nu} < U \psi_{\nu^*}$.
Lemma 9 then guarantees that for every \( \nu \in \text{Lim}(D) \cap S^3_2 \), \( h_\nu \) is the least upper bound of \( \{ \kappa^\nu_{(+)} \mid \tau \in D \cap \nu \} \) in the ordering \( <_U \). Since clearly \( \kappa^\nu_{(+)} \geq_U \kappa^\tau_{(+)} \) for every \( \tau \in D \cap \nu \), then

\[
(11) \quad h_\nu \leq_U \kappa^\nu_{(+)} \text{ for every } \nu \in \text{Lim}(D) \cap S^3_2
\]

We will eventually show that whenever \( R \) is a stationary subset of \( D \cap S^3_2 \), there are many \( \nu \in R \) such that \( h_\nu =_U \kappa^\nu_{(+)} \).

Also, we note that whenever \( \nu < \nu' < \nu'' \) are all in \( D \), then

\[
(12) \quad h_\nu <_U \kappa^\nu_{(+)} <_U h_{\nu''}.
\]

To see (12): since \( \nu^* \leq \nu' \) and \( (\nu')^* \leq \nu'' \) the bounding construction yields \( h_\nu <_U \kappa^\nu_{(+)} \leq U h_{\nu'} \leq U h_{\nu''} \).

**Lemma 13.** For every \( \nu \in \text{Lim}(D) \cap S^3_2 \):

1. There are \( U \)-many \( \alpha \) such that \( \theta^\nu_\alpha \) is a limit ordinal.
2. Let \( t_\nu = \langle \nu_\delta \mid \delta < \omega_2 \rangle \) be a sequence of members of \( D \) which is cofinal in \( \nu \). Then there are \( U \)-many \( \alpha \) such that
   a. \( h_\nu(\alpha) = \sup_{\delta < \alpha} \kappa^\nu_\delta \)
   b. \( \text{height}(N^\nu_\alpha) < h_\nu(\alpha) \) for all sufficiently large \( \delta < \alpha \).

**Proof.** First prove part 1. Fix any \( t_\nu = \langle \nu_\delta \mid \delta < \omega_2 \rangle \) which is a sequence of members of \( D \) which is cofinal in \( \nu \). Recall that the \( K^\nu_\alpha \) side of the coiteration is trivial and \( \theta^\nu_\alpha = \sup_{\delta < \alpha} \theta^\nu_\delta \) is the least stage of the \( K \) vs. \( K^\nu_\alpha \) coiteration where the iteration index is at least \( \text{ht}(\kappa^\nu_\alpha) \) (or the length of the \( K \) vs. \( K^\nu_\alpha \) coiteration, if there is no such index). For each \( \delta < \omega_2 \), let \( B_\delta := \{ \alpha \mid \theta^\nu_\alpha < \theta^\nu_{\delta+1} \} \); this is an element of \( U \) by (8) (note the least element of \( D \) above \( \nu_\delta \) is \( \leq \nu_{\delta+1} \)). Then \( B := \{ \alpha \mid \alpha \in \bigcap_{\delta \in B_\delta} \} \) is an element of \( U \) by Lemma 5. Then if \( \alpha \in B \), the sequence \( \langle \theta^\nu_\alpha \mid \delta < \alpha \rangle \) does not stabilize; so \( \theta^\nu_\alpha = \sup_{\delta < \alpha} \theta^\nu_\delta \) is a limit ordinal.

The proof of the part 2 is similar. Fix any \( \nu_\delta \mid \delta < \omega_2 \rangle \) which is contained in \( D \) and cofinal in \( \nu \). By (12), the set \( B_\delta := \{ \alpha \mid h_{\nu_\delta}(\alpha) < \kappa^\nu_{\delta+1} < h_{\nu_{\delta+2}}(\alpha) \} \) is an element of \( U \). By Lemma 5, \( B := \{ \alpha \mid \alpha \in \bigcap_{\delta \in B_\delta} \} \) is an element of \( U \). Pick any \( \alpha \in B \); then \( h_{\nu_\delta}(\alpha) < \kappa^\nu_{\delta+1} < h_{\nu_{\delta+2}}(\alpha) \) for cofinally many \( \delta < \alpha \), so \( \sup_{\delta < \alpha} h_{\nu_\delta}(\alpha) \leq \sup_{\delta < \alpha} \kappa^\nu_{\delta+1} \leq \sup_{\delta < \alpha} \kappa^\nu_{\delta+2}(\alpha) \). But the left and right sides of this last inequality are both \( h_\nu(\alpha) \), so \( h_\nu(\alpha) = \sup_{\delta < \alpha} \kappa^\nu_\delta \).

Finally, we find \( U \)-many \( \alpha \) such that \( \text{height}(N^\nu_\alpha) < h_\nu(\alpha) \) for sufficiently large \( \delta < \alpha \). Let \( \delta < \omega_2 \). The definition of \( \psi_{\nu_\delta} \) and the bounding construction imply \( \text{height}(N^\nu_\delta) \leq_U \psi_{\nu_\delta} < U h_{\nu_{\delta+1}} \); and since \( h_{\nu_\delta} <_U h_{\nu} \) (in fact \( h_{\nu_\delta}(\alpha) < h_{\nu}(\alpha) \) for all but nonstationarily many \( \alpha \in S^3_2 \)), in particular \( E_\delta := \{ \alpha \mid \text{height}(N^\nu_\alpha) < h_\nu(\alpha) \} \) is an element of \( U \). By
Lemma 5 there are $U$-many $\alpha$ such that $\alpha \in E_{\delta}$ for cofinally many $\delta < \alpha$. Then for such $\alpha$ we have $\text{height}(N^{\nu}_{\alpha}) < h_{\nu}(\alpha)$ for cofinally many $\delta < \alpha$. Since $\theta^{\nu}_{\alpha} = \sup_{\delta < \alpha} \theta^{\nu}_{\alpha}$ is a limit ordinal (by part 1), then in fact $\text{height}(N^{\nu}_{\alpha}) < h_{\nu}(\alpha)$ for all sufficiently large $\delta < \alpha$. \hfill \Box

So (11) and Lemma 13 imply that for all but nonstationarily many $\nu \in S_{2}^{3}$, if $\langle \nu_{\delta} | \delta < \omega_{2} \rangle$ is cofinal in $\nu$ then there are $U$-many $\alpha$ such that:

\begin{equation}
\sup_{\delta < \alpha} \kappa^{\nu}_{\alpha} = h_{\nu}(\alpha) \leq \kappa^{\nu}_{\alpha}
\end{equation}

We will eventually replace the inequality in (13) with an equality (mod $U$).

Much of the theory in the remainder of this section is implicit in [6]. A subtle but important difference, however, is that in the current paper we arrange that certain properties hold on $V$-clubs of ordinals in $S_{2}^{3}$, rather than just on stationary sets; this is essential to the proof in the later sections.

**Definition 14.** A system of objects from the winning mouse is a sequence of functions $S = \langle \text{Obj}^{\nu}_{\alpha}(\cdot) | \nu \in R \rangle$ where $R \subseteq S_{2}^{3}$ is stationary and for every $\nu \in R$: $\text{dom}(\text{Obj}^{\nu}_{\alpha}(\cdot)) \in U$ and $\text{Obj}^{\nu}_{\alpha}(\cdot) \in U$. $N^{\nu}_{\alpha}$.

**Lemma 15.** Suppose $R \subseteq \text{Lim}(D) \cap S_{2}^{3}$ and $S = \langle \text{Obj}^{\nu}_{\alpha}(\cdot) | \nu \in R \rangle$ is a system of objects from the winning mouse. Then there is a stationary $R' \subseteq R$, an ordinal $\bar{\tau} < \omega_{3}$ such that: for every $\nu \in R'$ there are $U$-many $\alpha$ where

\begin{enumerate}
  \item There is no truncation at any coiteration stages in the interval $[\theta^{\nu}_{\alpha}, \theta^{\nu}_{\alpha})$
  \item $\text{Obj}^{\nu}_{\alpha}$ is in the range of the iteration map $\pi^{\tau, \nu}_{\alpha}$ (see Definition 12 for the meaning of $\pi^{\tau, \nu}_{\alpha}$).
  \item The preimage of $\text{Obj}^{\nu}_{\alpha}$ by $\pi^{\tau, \nu}_{\alpha}$ appears in the corresponding mouse strictly below level $h_{\tau}(\alpha)$ of that mouse.
\end{enumerate}

**Proof.** For each $\nu \in R$ fix some $t_{\nu} = \langle \nu_{\delta} | \delta < \omega_{2} \rangle$ which is a sequence of points in $D$ which is cofinal in $\nu$. By Lemma 13, $\theta^{\nu}_{\alpha}$ is a limit ordinal for every $\nu \in D \cap S_{2}^{3}$ (for $U$-many $\alpha$); so there is some $\delta^{\nu}(\alpha) < \alpha$ such that a thread to $\text{Obj}^{\nu}_{\alpha}$ appears by stage $\theta^{\nu}_{\alpha}$ and there are no truncations for stages in the interval $[\theta^{\nu}_{\alpha}, \theta^{\nu}_{\alpha})$. More precisely: letting $\bar{\theta} := \theta^{\nu}_{\alpha}$ then $\text{Obj}^{\nu}_{\alpha}$ is in the range of the iteration map $\pi^{X, \nu}_{\alpha}$ whenever $X \in S^{\nu}$ and $\alpha = \alpha_{X}$. By the weak normality of $U$, for each $\nu \in R$ there is a $\delta^{\nu} < \omega_{2}$ and a set $A_{\nu} \in U$ such that $\delta^{\nu}_{\alpha} \leq \delta^{\nu}$ for every $\alpha \in A_{\nu}$; so the thread to $\text{Obj}^{\nu}_{\alpha}$ appears by stage $\theta^{\nu}_{\alpha}$ for every
\( \alpha \in A_t. \) The point is that for \( \alpha \in A_t, \) the superscript on the stage \( \theta^{\alpha^*}_{\alpha^*} \) no longer depends on \( \alpha. \)

Since \( R \subseteq \omega_3 \) is stationary and \( \delta^\nu \in \omega_2, \) there is a stationary \( \tilde{R} \subset R \) and a fixed \( \delta \) such that \( \delta^\nu = \delta \) for every \( \nu \in \tilde{R}. \) In other words, for every \( \nu \in \tilde{R} \) and every \( \alpha \in A_t, \) a thread to \( \text{Obj}^\nu_\alpha \) appears by stage \( \theta^{\alpha^*}_{\alpha^*}. \)

So for each \( \nu \in \tilde{R} \) define the following functions with domain \( A_t: \)

\[
\tilde{\theta}^\nu_\alpha := \theta^{\nu^*_\alpha} \\
\phi^\nu(\alpha) := \text{the preimage of } \text{Obj}^\nu_\alpha \text{ by } \pi^\nu_{\tilde{\theta}^\nu_\alpha, \theta^\nu_\alpha}
\]

Lemma 13—specifically the part about \( \text{height}(N^{\nu^*}_2) \)—implies for every \( \nu \in \tilde{R}: \)

\[
\phi^\nu <_U h^\nu
\]

Lemma 9 and (16) imply that for every \( \nu \in \tilde{R} \) there is a \( \tau(\nu) < \nu \) such that \( \phi^\nu <_U h_{\tau(\nu)}. \) So an application of the Fodor Lemma yields a fixed \( h^\nu \) and a stationary \( R' \subset \tilde{R} \) such that \( \phi^\nu <_U h^\nu \) for every \( \nu \in R'. \)

\[\text{Corollary 16. For all but nonstationarily many } \nu \in D \cap S_3^3: \text{ There are } U\text{-many } \alpha \text{ such that there are no truncations at any stage in the interval } [\theta^\nu_\alpha, \theta^{\nu^*}_\alpha]. \]

\[\text{Proof. Suppose to the contrary that there is a stationary } R \subseteq \text{Lim}(D) \cap S_3^3, \text{ such that for each } \nu \in R \text{ there are } U\text{-many } \alpha \text{ with some truncation in the interval } [\theta^\nu_\alpha, \theta^{\nu^*}_\alpha]; \text{ let } A_\nu \subseteq U \text{ be the collection of such } \alpha. \text{ Let } R' \subset R \text{ and } \tilde{\tau} < \omega_3 \text{ be as in the conclusion of Lemma 15. Pick any pair } \tilde{\nu} < \nu \text{ which are both in } R' \text{ and such that } \tilde{\nu} < \tilde{\nu}^* < \nu \text{ (recall the star superscript indicates the next element of } D \text{ above the point). Let } B_\nu \subseteq U \text{ be a collection of } \alpha \text{ where } (14) \text{ holds. Now } \tilde{\tau} < \tilde{\nu} < \tilde{\nu}^* < \nu \text{ and these are all in } D, \text{ so by } (8) \text{ the set } C := \{ \alpha | \theta^\nu_\alpha < \theta^{\nu^*_\alpha} < \theta^{\nu^*_\alpha} \} \text{ is an element of } U. \text{ So } A_\nu \cap B_\nu \cap C \subseteq U; \text{ pick any } \alpha \text{ in this intersection. Since } \alpha \in B_\nu \text{ then there are no truncations in } [\theta^\nu_\alpha, \theta^{\nu^*_\alpha}). \text{ But this contradicts that } \alpha \in A_\nu \cap C. \]

\[\text{Definition 17. Let } S = \{ \text{Obj}^\nu_\alpha | \nu \in R \} \text{ be a system of objects from the winning mouse. Let } R' \subset R. \text{ We say that } S \text{ lines up on } R' \text{ iff for every } \tilde{\nu} < \nu \text{ in } R': \]

\begin{itemize}
  \item There are \( U\)-many \( \alpha \in \bar{S}^\nu \cap \tilde{S}^\nu \) such that there are no truncations at any coiteration stages in the interval \( [\theta^\nu_\alpha, \theta^{\nu^*_\alpha}] \)
  \item \( \tilde{\theta}^\nu(\_\_) <_U \theta^\nu(\_\_) \)
  \item \( \text{Obj}^\nu(\_\_) \subseteq U \text{ range}(\pi^\nu_\alpha) \)
  \item \( \pi^\nu_\alpha(\text{Obj}^\nu(\_\_)) =_U \text{Obj}^\nu(\_\_) \)
\end{itemize}
If we specify that $\text{Obj}_\alpha := \kappa_\alpha$—i.e. if the system of objects from the winning mouse are chosen to be the first critical point beyond the $K$ vs. $K_{\alpha}^\nu$ coiteration—for every $\nu$ and $\alpha$, then as in [6] these objects line up nicely (Lemma 18 below). Later, in Section 6 under the additional assumption that $|^{\omega_2} \omega_1 / U|$ is small, we will choose the objects of interest to be extenders, and use the small ultrapower assumption to get the objects to line up nicely.

**Lemma 18.** Same assumptions as Lemma 15. Let $R' \subset R$ be the stationary set from the conclusion of that Lemma.

If the system of objects from the winning mouse are just the critical points (i.e. $\text{Obj}_{\nu}(\alpha) = \kappa_{\nu}(\alpha)$ for each $\nu \in R$) then there is a stationary $R'' \subseteq R'$ such that the system lines up on $R''$.

**Proof.** First define the function $\tilde{\phi}^{\nu}$ as follows; it is similar to the function $\phi^{\nu}$ from the proof of Lemma 15, but this new definition takes advantage of the fixed ordinal $\bar{\tau}$ obtained in that lemma:

\[\bar{\phi}^{\nu}(\alpha) := \text{the preimage of } \kappa^{\nu}(\alpha) \text{ by } \pi^{\nu}_{\theta_\bar{\tau}}\]

Then for every $\nu < \nu'$ in $R'$: $\tilde{\phi}^{\nu} \leq_U \tilde{\phi}^{\nu'}$; in fact:

\[(18) \text{ The collection of } \alpha \text{ such that } \tilde{\phi}^{\nu}(\alpha) > \tilde{\phi}^{\nu'}(\alpha) \text{ is non-stationary} \]

since whenever $X \in S^{\nu} \cap S^{\nu'}$ then the coiteration of $K$ with $K_{\alpha_X}^{\nu}$ is an “initial segment” of the coiteration of $K$ with $K_{\alpha_X}^{\nu'}$, so $\tilde{\phi}(\alpha_X) \leq \tilde{\phi}(\alpha_X)$. This uses the fact that the objects of interest are the critical points of the coiteration; it is where the proof breaks down if the objects of interest are, e.g., extenders (though in Section 6 we will do exactly that, with an additional assumption on $U$).

Then:

\[(19) \langle \tilde{\phi}^{\nu}|\nu \in R' \rangle \text{ is eventually constant modulo } U \]

To see (19): suppose this failed. If $\nu < \nu'$ are in $R'$ and $\tilde{\phi}^{\nu} \neq_U \tilde{\phi}^{\nu'}$, then by (18) and the fact that $U$ extends the club filter, the only possibility is that $\tilde{\phi}^{\nu} <_U \tilde{\phi}^{\nu'}$. So the failure of (19) implies there is an unbounded $\bar{R}' \subset R'$ (not necessarily stationary) such that $\langle \tilde{\phi}^{\nu}|\nu \in \bar{R}' \rangle$ is a $<_U$-increasing sequence.

For each $\nu \in \bar{R}'$ let $\check{\nu}$ denote the least element of $\bar{R}'$ above $\nu$. Let $\tilde{\phi}^{\nu} := \tilde{\phi}^{\nu} \upharpoonright \{ \alpha \in \text{dom}(\tilde{\phi}^{\nu}) \cap \text{dom}(\tilde{\phi}) | \tilde{\phi}^{\nu}(\alpha) < \tilde{\phi}(\alpha) \}$ (note then \text{dom}(\tilde{\phi}^{\nu}) \in U). Then $\tilde{\phi}^{\nu} <_U \tilde{\phi}$ and (18) implies that whenever $\nu < \nu'$ are in $\bar{R}'$ then $\tilde{\phi}^{\nu}$ and $\tilde{\phi}^{\nu'}$ agree at only nonstationarily many points.
Furthermore, there is a fixed function from \( \omega_2 \to \omega_2 \)—namely, \( h_{x} \)—which \( U \)-bounds \( \tilde{\phi}_\nu \) for every \( \nu \in \tilde{R}' \). This contradicts Corollary 7, and completes the proof of (19).

Let \( R'' \) be the tail end of \( R' \) on which \( \tilde{\phi}_\nu \) are pairwise equal modulo \( U \). Then \( R'' \) is the desired set.

**Lemma 19.** Assume \( R \subseteq S_2^3 \) is stationary. Let \( R'' \) be a stationary subset of \( R \) on which \( (\kappa'_\nu \mid \nu \in R) \) lines up (such an \( R'' \) exists from the conclusion of Lemma 18). Then \( \kappa'_\nu =_U h_\nu \) for every \( \nu \in R'' \cap \text{Lim}(R'') \).

**Proof.** Fix any sequence \( t_\nu = \langle \nu_\delta \mid \delta < \omega_2 \rangle \) of points in \( R'' \) which is cofinal in \( \nu \). Let \( A_\nu \subseteq U \) be a set which witnesses the conclusion of Lemma 13. For each \( \delta < \omega_2 \), let \( \nu_\delta, \nu \) be both in \( R'' \) and the system lines up on \( R'' \), there is a set \( A_{\nu_\delta, \nu} \subseteq U \) which witnesses this fact (so \( \pi_{\alpha, \nu_\delta, \nu}(\kappa'_\nu) = \kappa'_\nu \) for every \( \alpha \in A_{\nu_\delta, \nu} \)). By Lemma 5 the set \( \{ \alpha \mid A_{\nu_\delta, \nu} \cap A_\nu \} \) is an element of \( U \); for such \( \alpha \), let \( I_\alpha := \{ \delta < \alpha \mid \pi_{\alpha, \nu_\delta, \nu}(\kappa'_\nu) = \kappa'_\nu \} \). Then \( \kappa'_\nu = \pi_{\alpha, \nu_\delta, \nu}(\kappa'_\nu) = \sup_{\delta < \alpha} \kappa'_\nu \) for each \( \delta \in I_\alpha \) (since \( I_\alpha \) is cofinal in \( \alpha \). Since \( \alpha \in A_\nu \) then \( \sup_{\delta < \alpha} \kappa'_\nu = h_\nu(\alpha) \). So \( \kappa'_\nu = h_\nu(\alpha) \).

In particular, since \( R \) was assumed to be any stationary subset of \( S_2^3 \) then:

**Corollary 20.** \( \kappa'_\nu =_U h_\nu \) for all but nonstationarily many \( \nu \in S_2^3 \).

**Corollary 21.** Suppose for each \( \nu < \omega_3 \), \( b'_\nu \) is a subset of \( \nu \) which is an element of \( K \). Recall from Definition 2 that \( b'_\alpha \) is defined as \( \sigma_{x_\rho}(b') \) where \( x \) is an element of \( S_b' \) such that \( \alpha = \alpha_x \) and \( b' \in X \). Then \( b'_\nu \subseteq N'_\nu \) for all but nonstationarily many \( \nu \in S_2^3 \).

**Proof.** Since \( \nu' \in \varphi^K(\nu) \) then \( b'_\nu \subseteq (h_\nu(\nu)) = (K'\nu(\nu)) \); the last equality is by Corollary 20. Since the \( K'\nu \) side of the coiteration with \( K \) is simple (in fact trivial), then \( \varphi^K(\nu) \subseteq \varphi^K(\nu) \). So \( b'_\nu \subseteq (h_\nu(\nu)) \).

**Corollary 22.** Assume \( b \in \varphi^K(\omega_3) \) and \( R \) is a stationary subset of \( S_2^3 \). Let \( b'_\nu := b \cap \nu \) for each \( \nu < \omega_3 \). By Corollary 21, letting \( R_b := \{ \nu \in S_2^3 \mid b'_\nu \subseteq N'_\nu \} \), \( R - R_b \) is nonstationary, so \( S_b := \{ (\kappa'_\nu, b'_\nu) \mid \nu \in R_b \} \) is a system of objects from the winning mouse.

Then there is a stationary \( R'' \subseteq R_b \) on which \( S_b \) lines up.

**Proof.** Let \( R'_b \) be a stationary subset of \( R_b \) whose existence is guaranteed by Lemma 15, and let \( \bar{\tau} \) be the fixed ordinal from the conclusion of that lemma; so \( b'_\nu \subseteq (\pi_{\bar{\tau}}' \nu) \) for every \( \nu \in R'_b \).
Using Lemma 18, refine $R_b'$ further to a stationary $R_b''$ on which the critical points align; i.e. such that $\langle \kappa_{(\cdot)^{\alpha}}^{\nu} \rangle | \nu \in R_b'' \rangle$ lines up on $R_b''$.

Pick any $\bar{\nu} < \nu$ both in $R_b''$; WLOG assume $\bar{\tau} < \min(R_b'')$. Since $R_b'' \subset R_b'$ and $\bar{\tau} < \bar{\nu} < \nu$, then

$$b_{\alpha}^\nu \in \text{range}(\pi_{\alpha}^{\bar{\nu},\nu}) \text{ for } U\text{-many } \alpha.$$ Let $A_{\bar{\nu},\nu} \in U$ be the collection of such $\alpha$.

Now consider any $X \in S^\nu \cap S^\nu \cap S^b$, and let $\sigma_X : H_X \rightarrow H_\theta$ be the inverse of the Mostowski collapsing map for $X$. Let $\alpha = X \cap \omega_2$. Then

$$b_\alpha^\nu = b_{X}^{\nu} = \sigma_X^{-1}(b_{\nu}^{\nu}) = \sigma_X^{-1}(b_{\nu}^{\nu} \cap \bar{\nu}^{\nu}) = \sigma_X^{-1}(b_{\nu}^{\nu} \cap \sigma_X^{-1}(\bar{\nu}^{\nu})) = b_{\alpha}^{\nu} \cap h_{\nu}(\alpha)$$

Let $B_{\bar{\nu},\nu}$ be the collection of such $\alpha$; note $B_{\bar{\nu},\nu}$ is almost all of $S^\nu_1$ and is thus an element of $U$.

Let $C_{\bar{\nu},\nu} \in U$ be the collection of $\alpha$ such that

$$\kappa_\alpha^\nu = h_{\nu}(\alpha) \text{ and } \kappa_{\alpha}^{\nu} = h_{\nu}(\alpha)$$

(Note that by Corollary 20 we can WLOG assume that $\kappa_{(\cdot)^{\alpha}}^{\nu} = \nu \text{ for every } \tau \in R_b'$.)

Finally, let $D_{\bar{\nu},\nu}$ be the collection of $\alpha$ such that $\pi_{\alpha}^{\bar{\nu},\nu}(\kappa_{\alpha}^{\nu}) = \kappa_{\alpha}^{\nu}$; this set is in $U$ because $\langle \kappa_{(\cdot)^{\alpha}}^{\nu} \rangle | \nu \in R_b'' \rangle$ lines up on $R_b''$.

Then $\pi_{\alpha}^{\bar{\nu},\nu}(b_{\alpha}^{\nu}) = b_{\alpha}^{\nu}$ for any $\alpha \in A_{\bar{\nu},\nu} \cap B_{\bar{\nu},\nu} \cap C_{\bar{\nu},\nu} \cap D_{\bar{\nu},\nu}$. 

\hspace{2cm} $\square$

**Corollary 23.** Same assumptions as Corollary 22; let $R_b''$ be as in the conclusion of that corollary. Then for every $\nu \in R_b'' \cap \text{Lim}(R_b'')$ and every sequence $s_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ of points in $R_b''$ which is cofinal in $\nu$, there is a set $G_{b_\nu}^\nu \in U$ such that for every $\alpha \in G_{b_\nu}^\nu$:

$$h_{\nu}(\alpha) = \kappa_{\alpha}^{\nu} \text{ and for cofinally many } \delta < \alpha:\n
(\nu_\delta(\alpha) = \kappa_{\alpha}^{\nu} \text{ and for cofinally many } \delta < \alpha):$$

$$\bullet \quad h_{\nu_\delta}(\alpha) = \kappa_{\alpha}^{\nu_\delta}\n
\bullet \quad \pi_{\alpha}^{\nu_\delta,\nu}(\kappa_{\alpha}^{\nu_\delta}) = \kappa_{\alpha}^{\nu}\n
\bullet \quad \pi_{\alpha}^{\nu_\delta,\nu}(b_{\alpha}^{\nu_\delta}) = b_{\alpha}^{\nu}$$

**Proof.** Fix a $\nu \in R_b'' \cap \text{Lim}(R_b'')$ and cofinal sequence $s_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ ($\subset R_b''$) as in the statement of the corollary. For each $\delta < \omega_2$ let $G_{b_\nu}^\nu$ be the collection of $\alpha$ such that:

$$\bullet \quad h_{\nu_\delta}(\alpha) = \kappa_{\alpha}^{\nu_\delta}\n
\bullet \quad \pi_{\alpha}^{\nu_\delta,\nu}(\kappa_{\alpha}^{\nu_\delta}) = \kappa_{\alpha}^{\nu}\n
\bullet \quad \pi_{\alpha}^{\nu_\delta,\nu}(b_{\alpha}^{\nu_\delta}) = b_{\alpha}^{\nu}$$

Then $G_{b_\nu}^\nu$ is an element of $U$ since the system $S_\delta$ lines up on $R_b''$. By Lemma 5 there are $U$-many $\alpha$ such that $\alpha \in G_{b_\delta}$ for cofinally many $\delta < \alpha$. $\square$
If we make the additional assumption on $U$ that $|ω^2ω_1/U| = ω_2$, then every system of objects from the winning mouse lines up on a stationary set:

**Lemma 24.** Assume $U$ is a weakly normal ultrafilter concentrating on $S^2_1$ and $|ω^2ω_1/U| = ω_2$. Then for every system $S = \langle Obj^-\nu | \nu \in R \rangle$ of objects from the winning mouse, there is a stationary $R'' \subset R$ on which $S$ lines up.

**Proof.** Let $R'$ and $\bar{\tau} < ω_3$ be the stationary set and ordinal, respectively, guaranteed by Lemma 15. For each $\nu \in R'$ define

$$ \bar{\phi}^\nu(\alpha) = \text{the preimage of Obj}^\nu \text{ by the iteration map } \pi^{\bar{\tau},\nu}. $$

Recall by Lemma 15 that for every $\nu \in R'$, there are $U$-many $\alpha$ such that the level of the mouse $N^\nu_\alpha$ at which $\bar{\phi}^\nu(\alpha)$ appears is strictly less than $h_{\bar{\tau}}(\alpha)$. For each transitive $N \in H_{ω_3}$ fix an injection $g_N : N \rightarrow ω_1$. Let $N(\bar{\tau},\alpha)$ be the initial segment of the mouse $N^\nu_\alpha$ corresponding to level $h_{\bar{\tau}}(\alpha)$. Define

$$ \psi^\nu(\alpha) := g_{N(\bar{\tau},\alpha)}(\bar{\phi}^\nu(\alpha)) $$

Note $ψ^\nu(\alpha)$ is defined for $U$-many $\alpha$; so each $ψ^\nu$ is a function whose domain is in $U$ and maps into $ω_1$. Since $|ω^2ω_1/U| = ω_2$ and $R'$ is a stationary subset of $ω_3$, then by the $ω_3$-completeness of $NS_{ω_3}$ there is a stationary $R'' \subset R'$ such that $ψ^\nu =_U ψ^\nu$ for every $\bar{\nu}, \nu$ in $R''$. It follows that $\bar{\phi}^\nu =_U \bar{\phi}^\nu$ for every $\bar{\nu}, \nu$ in $R''$, and so $S$ lines up on $R''$. □

For the rest of the paper, $D'$ will refer to the collection of $\nu \in D \cap S^3_2$ such that $κ^\nu_{(-)} =_U h_\nu$; by Corollary 20 $D'$ is almost all of $S^3_2$.

**5. Proof of part 1 of Theorem 1**

As in Chapter 8 of [14], for mice below 0-pistol, $o^M_*(κ)$ denotes the “Mitchell order of $κ$ in $M$”; more precisely, the ordertype of the collection of $ν \geq κ^+M$ which index an extender with critical point $κ$. $o^M(κ)$ denotes the least primitively recursively closed ordinal which is at least $κ^+$ and does not index an extender with critical point $κ$. If $E$ is an extender on the $M$ sequence which has critical point $κ$ and is generated by a single normal measure of $M$-Mitchell order $λ$, we will use $U^M(κ, λ)$ to denote this normal measure. For the rest of the paper, our background assumption is that there are no extenders on any mice with two generators; so every extender is generated by such a normal measure.
Theorem 25. Let $b \in \wp^K(\omega_3)$ code a wellordering of $\omega_3$. Then $o^K_\nu(b) > \text{otp}(b \cap \nu)$ for all but nonstationarily many $\nu \in S^3_2$ (“all but nonstationarily many” is in the sense of $V$).

This could be equivalently formulated in terms of canonical functions on $\omega_3$, but we will avoid that formulation to avoid confusion with the canonical functions on $\omega_2$ which are already in use. We also find a nice characterization of the measures on such $\nu$ (Lemma 28). This characterization will then be used to define a $K$-ultrafilter on $\omega_3$ and show it has the desired properties.\footnote{One of these properties is that the Mitchell order of this $K$-ultrafilter will be $\text{otp}(b)$. Since $b$ was an arbitrary wellorder of $\omega_3$ in $K$, this would imply that $o^K_\kappa(\kappa) \geq \kappa^+K$, where $\kappa = \omega^K_\nu$.}

Let $\gamma := \text{otp}(b)$; so $\gamma < \omega^+3$. We assume by induction that Theorem 25 holds for all ordinals $< \gamma$; i.e. whenever $\tau < \gamma = \text{otp}(b)$ then there is an $\omega_2$-club $C_{>\tau} \subset \omega_3$ in $V$ such that $o^K_\nu(b) > \text{otp}(b \cap \nu)$ for every $\nu \in C_{>\tau}$, where $b_\tau \in \wp^K(\omega_3)$ has ordertype $\tau$.

Claim 26. There is an $\omega_2$-club $C_{\geq b} \subset \omega_3$ such that $o^K_\nu(b) \geq \text{otp}(b \cap \nu)$ for every $\nu \in C_{\geq b}$.

Proof. Fix some sequence $\langle \tau_i | i < \text{cf}(\gamma) \rangle \in V$ cofinal in $\gamma$; if $\gamma$ is a successor ordinal, say $\gamma = \tau_0 + 1$, the sequence is just $\langle \tau_0 \rangle$. Fix a large regular $\theta > 2^{\omega_3}$ and consider the collection $\hat{C}$ of all $Z \in P_{\omega_3}(H_\theta)$ such that $Z \prec (H_\theta, \langle (\tau_i, C_{>\tau_i}, b_\tau_i)|i < \text{cf}(\gamma) \rangle, ..., Z \cap \omega_3 \in S^3_2$. Let $C_{\geq b} := \{Z \cap \omega_3 | Z \in \hat{C}\}$. For $Z \in \hat{C}$ let $\pi_Z : H_Z \rightarrow H_\theta$ be the inverse of the Mostowski collapse of $Z$. Let $\nu_Z := cr(\pi_Z) = Z \cap \omega_3$.

Now for all $i \in Z \cap \omega_3 = \nu_Z$, $\nu_Z \in C_{>\tau_i}$.\footnote{Because $C_{>\tau_i} \in Z$ and so (by elementarity) $\nu_Z$ is a limit of $C_{>\tau_i}$; since $\nu_Z$ has cofinality $\omega_2$, and $C_{>\tau_i}$ is an $\omega_2$-club, then $\nu_Z \in C_{>\tau_i}$.}

So:

- if $\text{cf}(\gamma) = \omega_3$, then $\nu_Z \in \Delta_{i<\omega_3}C_{>\tau_i}$ and $\text{otp}(b \cap \nu_Z) = \sup_{i<\omega_2} \text{otp}(b_\tau_i \cap \nu_Z)$.
- if $\text{cf}(\gamma) < \omega_3$ is a limit ordinal, then $\nu_Z \in \cap_{i<\text{cf}(\gamma)}C_{>\tau_i}$ and $\text{otp}(b \cap \nu_Z) = \sup_{i<\text{cf}(\gamma)} \text{otp}(b_\tau_i \cap \nu_Z)$
- if $\gamma$ is a successor ordinal, say $\gamma = \tau + 1$, then $\nu_Z \in C_{>\tau}$ and $\text{otp}(b \cap \nu_Z) = \text{otp}(b_\tau \cap \nu_Z) + 1$.

In all 3 cases, $o^K_\nu(\nu_Z) \geq \text{otp}(b \cap \nu_Z)$.

Let $b^\nu := b \cap \nu$ for every $\nu < \omega_3$. Note that if $\nu \in D' \cap C_{\geq b}$, then $h_\nu(\alpha) = \kappa^\nu_\alpha$, $\mu^\nu_\alpha = o^K_\nu(\kappa^\nu_\alpha)$,\footnote{This follows from (5).} and the Mitchell order of $E^N_{\mu^\nu_\alpha}$ is at least $\text{otp}(b^\nu_\alpha)$ for $U$-many $\alpha$. Furthermore, by Corollary 16 there is no
truncation at stage $\theta_\alpha^\nu$, so the extender applied by the $K$ side at this stage is total in $N_\alpha^{\nu}$. In particular:

For every $\nu \in C_{\geq b}$ there are $U$-many $\alpha$ such that

\begin{equation}
N_\alpha^{\nu} \text{ has a total extender on } h_\nu(\alpha) \text{ of Mitchell order otp}(b_\alpha^{\nu}).
\end{equation}

So for every $\nu \in C_{\geq b}$ the following definition makes sense: define an ultrafilter $W^{b_\nu}$ on $\varphi^K(\nu)$ by

\[ z \in W^{b_\nu} \text{ iff } \{ \alpha \in S^\nu ; z_\alpha^{\nu} \in U^{N_\alpha^{\nu}}(h_\nu(\alpha), \text{otp}(b_\alpha^{\nu})) \} \in U. \]

(i.e. iff $\text{ult}(V, U) = [z^{\nu}]_U \in U^{[N_\alpha^{\nu}]_U}([h_\nu]_U, [b_\alpha^{\nu}]_U)$.) Here $b_\alpha^{\nu}$ and $z_\alpha^{\nu}$ are defined as $(\sigma_\alpha^{\nu})^{-1}(b)$ and $(\sigma_\alpha^{\nu})^{-1}(z)$ (respectively), for those $\alpha$ such that $b, z \in \text{rng}(\sigma_\nu^{\nu})$ (see Definition 2). For the remainder of the proof, to cut down on notation we will often just write $b_\alpha^{\nu}$ instead of $\text{otp}(b_\alpha^{\nu})$; e.g. $U^{N_\alpha^{\nu}}(k_\alpha^{\nu}, b_\alpha^{\nu})$ means the normal measure of Mitchell order $\text{otp}(b_\alpha^{\nu}).$

It is clear that $W^{b_\nu}$ is an ultrafilter on $\varphi^K(\nu)$. Our goals are to ultimately show that $W^{b_\nu}$ generates $K$’s extender of order $\text{otp}(b_\nu)$, and to characterize the $W^{b_\nu}$’s in a way that allows us to build a $K$-measure on $\omega_3$ of Mitchell order $\text{otp}(b)$.

**Lemma 27.** For all but nonstationarily many $\nu \in S^3_2$: $\sigma^K_\nu(\nu) > \text{otp}(b^{\nu}).$

**Proof.** Suppose this fails; then there is a stationary $R_b \subset C_{\geq b}$ such that $\sigma^K_\nu(\nu) = \text{otp}(b^{\nu})$ for every $\nu \in R_b$.

Since $b^{\nu} \in \varphi^K(\nu)$, then by Corollary 21, $S_b := \langle (b^{\nu}_{(-)}, k^{\nu}_{(-)}) \mid \nu \in R_b \rangle$ is a system of objects from the winning mouse. Since the $b^{\nu}$ are initial segments of $b$, by Corollary 22 there is a stationary $R_b'' \subset R_b$ on which $S_b$ lines up.

For the remainder of the proof of Lemma 27, fix some $\nu \in R_b'' \cap \text{Lim}(R_b'')$ and any $t_\nu = \langle \nu_\delta \mid \delta < \omega_2 \rangle$ which is a subset of $R_b''$ and cofinal in $\nu$.

**Claim 27.1.** $\langle \nu_\delta \mid \delta < \omega_2 \rangle$ generates $W^{b_\nu}$; i.e. $z \in W^{b_\nu}$ iff $\nu_\delta \in z$ for sufficiently large $\delta < \omega_2$.

(Note we are still proving Lemma 27 by contradiction; the current claim will not in general characterize the measures $W^{b_\nu}$.)

**Proof.** (of Claim 27.1) Let $z \in W^{b_\nu}$; so $z_{(-)} \in_U U^{N_\nu}_{(-)}(h_{(-)}(\cdot), b^{(-)}_{(-)})$ by definition. Recall $\theta_\alpha = \sup_{\delta < \alpha} \theta_\alpha^{\nu}$ is a limit ordinal for $U$-many $\alpha$. So for such $\alpha$ there is a $\delta(\alpha) < \alpha$ such that a thread to $z_\alpha$ appears by stage $\theta_\alpha^{\nu}(\alpha)$ of the $K$ vs. $K_\alpha^\nu$ coiteration. By weak normality of $U$, there is a $\delta$ such that for $U$-many $\alpha$, a thread to $z_\alpha$ appears by stage $\theta_\alpha^{\nu}(\alpha)$.

\begin{equation}
\text{For every } \delta \in [\delta, \omega_2); z_{(-)} \in_U \text{range}(\pi^{\nu}_{(-)}).\end{equation}
We will show \( \nu_3 \in z \) for every \( \delta \in [\delta, \omega_2) \): Fix such a \( \delta \). Recall that 
\[ \kappa^\nu_{(-)} = \mathcal{U} h_{\nu_3}(-) \] 
and 
\[ \kappa^\nu_{(-)} = \mathcal{U} h_{\nu}(-) \]. \( \nu_3 \) and \( \nu \) are both elements of \( R^\nu_b \) and \( \mathcal{S}_b \) lines up on \( R^\nu_b \), so 
\[ \pi^\nu_{(\alpha, \delta)}(b^\nu_\alpha, \kappa^\nu_{(\alpha)}) = (b^\nu_\alpha, \kappa^\nu_{(\alpha)}) \] 
for \( U \)-many \( \alpha \). And since \( z_{(-)} \in U \mathcal{U}^\nu_{(-)}(h_{\nu}(-), b^\nu_{(-)}) \) (by assumption), then for \( U \)-many \( \alpha \):

\[ z_{\alpha} \cap \kappa^\nu_{\alpha} = (\pi^\nu_{\alpha, \delta})^{-1}(z_{\alpha}) \in U \mathcal{U}^\nu_{\alpha}(\kappa^\nu_{\alpha}, b^\nu_\alpha \cap \kappa^\nu_{\alpha}) = \]

Since \( \nu_3 \) is in \( R^\nu_b \) then

\[ b^\nu_\alpha = \mathcal{U} \) the Mitchell order of \( h_{\nu_3}(-) \) in \( K^\nu_{(-)} \)

\[ = \mathcal{U} \) the Mitchell order of \( \kappa^\nu_{\alpha} \) in \( K^\nu_{\alpha} \)

So by (26) and (27), there are \( U \)-many \( \alpha \) such that \( z_{\alpha} \cap h_{\nu_3}(\alpha) \) is an element of the measure applied at stage \( b^\nu_\alpha \)—and so \( h_{\nu_3}(\alpha) \in z_{\alpha} \)—for \( U \)-many \( \alpha \). Pick one such \( \alpha \); then \( \sigma^\nu_{\alpha}(h_{\nu_3}(\alpha)) = \nu_3 \) is an element of \( \sigma^\nu_{\alpha}(z_{\alpha}) = z \).

This completes the proof that every element of \( W^{b^\nu} \) contains a tail end of the set \( \{ \nu_3 \delta < \omega_2 \} \). Since \( W^{b^\nu} \) is an ultrafilter on \( K(\nu) \), then the converse is also true. \( \square \)(Claim 27.1)

We will complete the proof of Lemma 27 by showing \( W^{b^\nu} \) is on \( K \)'s sequence and has Mitchell order \( \text{otp}(b^\nu) \); this will contradict that \( \nu \in R^\nu_b \).

**Claim 27.2.** \( \{ \xi < \nu | \alpha^K(\xi) = \text{otp}(b \cap \xi) \} \in W^{b^\nu} \)

**Proof.** This follows easily from the definition of \( W^{b^\nu} \) and the fact that each \( \mathcal{U}^\nu_{\alpha}(h_{\nu}(\alpha), b^\nu_\alpha) \) has Mitchell order \( b^\nu_\alpha \). \( \square \)

By Corollary 29 from [2], to see that \( W^{b^\nu} \) generates an extender on \( K \)'s extender sequence, it suffices\(^{11}\) to show that \( W^{b^\nu} \) is normal with respect to \( K \) and that \( \text{ult}(K[\nu^+K], W^{b^\nu}) \) is wellfounded. Let \( G^b_{s^\nu} \in U \) be the set from the conclusion of Corollary 23 (recall \( s_\nu = \nu_3 \delta < \omega_2 \) is some sequence which is contained in \( R^\nu_b \) and cofinal in \( \nu \)). Pick an \( X \in P^{\omega_2}(H_{\theta}) \) with \( W^{b^\nu} \in X < H_{\theta} \) and \( \alpha_X := X \cap \omega_2 \in G^b_{s^\nu} \); there is such an \( X \) because \( G^b_{s^\nu} \in U \) and is thus a stationary subset of \( S^2_{1} \). Let \( \sigma_X : H_X \rightarrow H_{\theta} \) be the inverse of the Mostowski collapsing map. Let \( W_X := \sigma_X^{-1}(W^{b^\nu}) \). By Claim 27.1:

\[ W_X \text{ is generated by } \sigma_X^{-1}(\langle \nu_3 \delta < \omega_2 \rangle) = \langle h_{\nu_3}(\alpha_X) \delta < \alpha_X \rangle \]

\(^{11}\)Under the assumption that 0-pistol does not exist, which we are assuming throughout.
Note that $\alpha^K_\nu(\nu) = b^{\nu+\nu}$ for every $\delta < \omega_2$, and so:

$$\alpha^K_\nu(h_\nu(\alpha_X)) = b^{\nu+\nu}_{\alpha_X} \text{ for every } \delta < \alpha_X.$$ (29)

Let $I$ be the collection of $\delta < \alpha_X$ such that (23) holds; since $\alpha_X \in G^b_s$ then $I$ is cofinal in $\alpha_X$. Then the definition of $I$ and (29) imply that the Mitchell order of the extender applied at stage $\alpha_X^{\nu+\nu}$ is equal to $\text{otp}(b^{\nu+\nu}_{\alpha_X})$, and $\pi^{\nu+\nu}_{\alpha_X}(b^{\nu+\nu}_{\alpha_X}) = b^{\nu+\nu}_{\alpha_X}$ for every $\delta \in I$. I.e. the measure $\mathcal{U}^{\nu+\nu}_{\alpha_X} := \mathcal{U}^{\nu+\nu}_{\alpha_X}(h_\nu(\alpha_X), b^{\nu+\nu}_{\alpha_X})$ in the iterate $N^{\nu+\nu}_{\alpha_X}$ is the result of “repeating a measure” at stages of the form $\alpha_X^{\nu+\nu}$ for $\delta \in I$; and the critical point at each such stage was of the form $h_\nu(\alpha_X)$. So

$$\text{The measure } \mathcal{U}^{\nu+\nu}_{\alpha_X} \text{ is generated by } \langle h_\nu(\alpha_X) | \delta \in I \rangle.$$ (30)

Since $\wp^K_X(h_\nu(\alpha_X)) \subseteq \wp^{\nu+\nu}_{\alpha_X}(h_\nu(\alpha_X))$, then (28) and (30) imply:

$$W_X = \mathcal{U}^{\nu+\nu}_{\alpha_X} \cap \wp^K_X(h_\nu(\alpha_X))$$ (31)

(In fact they’re equal; we could WLOG assume there is no truncation at stage $\alpha_X^{\nu+\nu}$ by Corollary 16.)

Now suppose that either

• $W^{\nu+\nu}$ were not normal with respect to $K$, or
• $\text{ult}(K|\nu^{+K}, W^{\nu+\nu})$ were illfounded.

Then $H_X$ would believe the same about $W_X$ with respect to $K_X$. We show how to achieve a contradiction if $\text{ult}(K|\nu^{+K}, W^{\nu+\nu})$ is illfounded; if $W^{\nu+\nu}$ fails to be normal with respect to $K$, the proof is similar. So suppose $\text{ult}(K|\nu^{+K}, W^{\nu+\nu})$ is illfounded. Then by elementarity of $\sigma_X$, $\text{ult}(K_X|h_\nu(\alpha_X)^{+K_X}, W_X)$ is illfounded; let $\{f_n | n \in \omega\} \in H_X$ witness this fact. Since $h_\nu(\alpha_X)$ is the largest cardinal in $K_X|h_\nu(\alpha_X)^{+K_X}$, then WLOG we assume $f_n : h_\nu(\alpha_X) \rightarrow h_\nu(\alpha_X)$. But $h_\nu(\alpha_X) = \kappa^{\nu+\nu}_{\alpha_X}$ and so $\wp^K_X(h_\nu(\alpha_X)) \subseteq \wp^{\nu+\nu}_{\alpha_X}(h_\nu(\alpha_X))$. By (31), $\{f_n | n \in \omega\}$ witnesses an illfounded chain in $\text{ult}(N^{\nu+\nu}_{\alpha_X}, \mathcal{U}^{\nu+\nu}_{\alpha_X})$, which is a contradiction because $\mathcal{U}^{\nu+\nu}_{\alpha_X}$ is on the extender sequence of the mouse $N^{\nu+\nu}_{\alpha_X}$. (In the case where we assume $W_X$ fails to be normal in $K_X$, we would contradict the fact that $\mathcal{U}^{\nu+\nu}_{\alpha_X}$ is normal with respect to $N^{\nu+\nu}_{\alpha_X}$.)

So $K$ has an extender of order $\text{otp}(b \cap \nu)$ for all but nonstationarily many $\nu \in S_3^2$. This concludes the proof of Theorem 25. Next we show that, for many $\nu$, the extender on $\nu$ of order $b^{\nu+\nu}$ in $K$ is exactly (the extender generated by) $W^{\nu+\nu}$. This fact will be used later in the definition of measures on $\omega_3$.

**Lemma 28.** Let $C_{>b}$ be an $\omega_2$-closed unbounded subset of $\nu$ such that $\alpha^K_\nu(\nu) > \text{otp}(b \cap \nu)$ for every $\nu \in C_{>b}$ (such a set exists by Lemma 27). Let $R_b$ be any stationary subset of $C_{>b}$. Then there is a stationary $R' \subset R_b$ with the properties:
• $W^{b'}$ is (generates) $K$’s extender with critical point $\nu$ of order $b'$ for every $\nu \in R'_b$.

• Whenever $\nu \in R'_b \cap \text{Lim}(R''_b)$ then $W^{b'}$ has the following characterization:

\[
z \in W^{b'} \text{ iff } z \cap \nu' \in U^K(\nu', b') \text{ for sufficiently large } \nu' \in \nu \cap R''_b.
\]

Proof. Let $S = \langle b''_\delta, \kappa''_\delta | \delta \in R_b \rangle$; by Corollary 21 $S$ is a system of objects of the winning mouse. By Corollary 22 there is a stationary $R''_b \subset R_b$ on which $S$ lines up.

Pick any $\nu \in R''_b \cap \text{Lim}(R''_b)$ and fix any sequence $t_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ of points in $R''_b$ which is cofinal in $\nu$. So for each $\delta < \omega_2$ we know $K$ has a measure of order $b''_\delta$ on $\nu_\delta$, but we don’t yet know that it is just $W^{b''_\delta}$. Similarly we know $K$ has a measure of order $b''_\delta$ on $\nu$ but do not yet know that it is $W^{b'}$.

The proof of (32) is very similar to the proof of Claim 27.1, but note that in the current proof $W^{b'}$ is being characterized by reflection rather than by a generating sequence. Very briefly, suppose we are given some $z \in W^{b'}$. Use weak normality of $U$ and the fact that $\theta''_\delta$ is a limit ordinal for $U$-many $\alpha$ to find a $\delta^* < \omega_2$ such that $z(-) \in U \text{ range}(\pi_{\nu, \nu'}^\nu)$ (and for $U$-many $\alpha$ there are no truncations at stages in the interval $[\theta''_\delta, \theta''_{\omega_2}]$). Then pick any $\delta \in [\delta^*, \omega_2)$. First, since $\nu_\delta \in R_b$ then $o^K(\nu_\delta) > b''_\delta$; this implies that for $U$-many $\alpha$, $K''_\alpha$ and $N''_\alpha$ have exactly the same measure of order $b''_\alpha$ on $h_\nu(\alpha)$, since the index of the measure applied on the $N''_\alpha$ side is just the least ordinal which does not index an extender on the $K''_\alpha$ side (see (5)). Moreover, by coherency of the iteration, $N''_\alpha$ and $N''_\alpha$ have the same measure of order $b''_\alpha$ on $h_\nu(\alpha)$. Finally, since the $K''_\alpha$ side of the coiteration is trivial, then $K''_\alpha$ is just an initial segment of $K''_\alpha$. To summarize: the mice $N''_\alpha$, $N''_\alpha$, and $K''_\alpha$ all have exactly the same measure of order $b''_\alpha$ on $h_\nu(\alpha)$; let $\nu_\delta$ denote this common measure. Then use the fact that $\nu_\delta, \nu$ are both in $R''_b$, along with the assumption that $z \in W^{b'}$, to show that $z_\alpha \cap h_\alpha(\alpha) \in \nu_\delta$; then applying $\sigma''_\alpha$ yields that $z \cap \nu_\delta \in U^K(\nu_\delta, b''_\delta)$. This completes the sketch of the proof of (32).

Finally, we use (32) to show that $W^{b'}$ is in fact on $K$’s extender sequence. The proof is similar to the end of the proof of Lemma 27. Let $G^{b'}_{t_\nu} \in U$ be the set from the conclusion of Corollary 23. Pick an $X \in P_{\omega_1}(H_\theta)$ of cardinality $\omega_1$ such that $t_\nu \in X$ and $\alpha_X \in G^{b'}_{t_\nu}$; again, this is possible because $G^{b'}_{t_\nu} \in U$ and is thus stationary. Let $\sigma_X : H_X \to H_\theta$ be the inverse of the Mostowski collapsing map of $X$, and let $W_X = \sigma_X^{-1}(W^{b'})$. Let $I$ be the collection of $\delta < \alpha_X$ where (23) holds; since $\alpha_X \in G^{b'}_{s_{\alpha_X}}$ then $I$ is cofinal in $\alpha_X$. Now for every $\delta \in I$, the
measure on $h_{\nu_\delta}(\alpha_X)$ of Mitchell order $b_{\alpha_X}^{\nu_\delta}$ is exactly the same in the mice $K_X$, $N_{\alpha_X}^{\nu_\delta}$, and $N_{\alpha_X}^{\nu}$; let $\mathcal{V}_\delta$ denote this common measure. Then the measure $U_{\alpha_X}^{\nu_\delta} := U^{N_{\alpha_X}}(h_{\nu_\delta}(\alpha_X), b_{\alpha_X}^{\nu_\delta})$ has the following characterization:

\begin{equation}
\tag{33}
z \in U_{\alpha_X}^{\nu_\delta} \iff z \cap h_{\nu_\delta}(\alpha_X) \in \mathcal{V}_\delta \text{ for all sufficiently large } \delta \in I.
\end{equation}

The elementarity of $\sigma_X$ and (32) imply:

\begin{equation}
\tag{34}
z \in W_X \iff z \cap h_{\nu_\delta}(\alpha_X) \in \mathcal{V}_\delta \text{ for sufficiently large } \delta < \alpha_X.
\end{equation}

Then (33) and (34) imply:

\begin{equation}
\tag{35}
W_X = U_{\alpha_X}^{\nu_\delta} \cap \wp^K_X(h_{\nu}(\alpha_X))
\end{equation}

Similarly to the proof of Lemma 27, (35) can be used to show that $W_X$ is normal with respect to $K_X$ and that $\text{ult}(K_X, W_X)$ is wellfounded. So $W_X$ generates an extender on $K_X$’s extender sequence, and so by elementarity of $\sigma_X$, $W^{b^\nu}$ generates an extender on $K$’s extender sequence (and (32) provided the desired characterization of $W^{b^\nu}$).

\[\square\text{(Lemma 28)}\]

Finally, we build the measures on $\omega_3$. Fix a stationary $R''_b$ as in the conclusion of Lemma 28. Although we will not use this fact, it is interesting to note that such an $R''_b$ can be obtained within any stationary subset of $S_3^2$ (see the statement of Lemma 28).

Define a filter $F^b$ on $\wp^K(\omega_3)$ by:

\[z \in F^b \text{ iff } z \cap \nu \in W^{b^\nu} \text{ for sufficiently large } \nu \in R''_b\]

**Claim 29.** $F^b$ is an ultrafilter on $\wp^K(\omega_3)$.

**Proof.** Suppose not; so there is a $z \in \wp^K(\omega_3)$ such that both $R''_{b,z} := \{ \nu \in R''_b \mid z \cap \nu \in W^{b^\nu} \}$ and $R''_{b,z'} := \{ \nu \in R''_b \mid z \cap \nu \in W^{b_{\nu}} \}$ are cofinal in $\omega_3$. At least one of $R''_{b,z}$, $R''_{b,z'}$ must be stationary; WLOG assume $R''_{b,z}$ is stationary. Then since $R''_{b,z'}$ is unbounded in $\omega_3$, there is a $\nu \in R''_{b,z} \cap \text{Lim}(R''_{b,z'})$. Fix a sequence $t_\nu = \langle \nu_\delta \mid \delta < \omega_2 \rangle$ of points in $R''_{b,z'}$ which is cofinal in $\nu$. Since $z \cap \nu \in W^{b^\nu}$ and $t_\nu \subseteq R''_b$, then Lemma 28 yields that $z \cap \nu_\delta \in W^{b^\nu}$ for sufficiently large $\delta < \omega_2$. But this contradicts that each $\nu_\delta$ is an element of $R''_{b,z'}$.

\[\square\]

Using the definition of $F^b$ (and the fact that each $W^{b^\nu}$ is on $K$’s extender sequence), it is easy to see that $F^b$ is normal with respect to $K$, $\text{ult}(K, F^b)$ is wellfounded, and $F^b$ concentrates on the set $\{ \xi < \omega_3 \mid \theta^K(\xi) = \text{otp}(b \cap \xi) \}$. By Corollary 29 from [2], $F^b$ generates $K$’s total extender on $\omega_3$ of Mitchell order $\text{otp}(b)$. 

6. Proof of part 2 of Theorem 1

In this section we make the additional assumption that \(|\omega_2 \omega_1/U| = \omega_2\) in order to show there is a mouse which has an extender with 2 generators. We keep all notation from the previous section; in particular \(D'\) is the set of \(\nu \in S_2^3\) such that \(h_\nu = \nu \kappa_\omega^-\); recall \(D'\) is almost all of \(S_2^3\).

For \(\nu \in D'\), by Lemma 13 there are \(U\)-many \(\alpha\) such that \(\theta^\nu_\alpha + 2 \leq \theta^\nu_{\alpha^*}\) and \(\theta^\nu_\alpha\) is a limit ordinal, among other properties; recall \(\nu^*\) is the least element of \(D\) above \(\nu\). For such \(\alpha\), let \(E^\nu_\alpha\) denote the extender applied at stage \(\theta^\nu_\alpha\); recall \(\mu^\nu_\alpha\) denotes the index of \(E^\nu_\alpha\) on \(N^\nu_\alpha\)'s extender sequence, and \(\mu^\nu_\alpha = o^K_\alpha(h_\nu(\alpha))\) (for \(U\)-many \(\alpha\)). For an ordinal \(\eta < \mu^\nu_\alpha = lh(E^\nu_\alpha)\), \((E^\nu_\alpha)_\eta\) denotes the \(N^\nu_\alpha\)-ultrafilter \(\{z | \eta \in E^\nu_\alpha(z)\}\) (i.e. viewing \(E^\nu_\alpha\) as a hypermeasure, it is the \(\eta\)-th element of the hypermeasure). If \(\eta\) is primitive recursively closed then \(E^\nu_\alpha|\eta\) denotes the extender \(z \mapsto E^\nu_\alpha(z) \cap \eta\).

For each \(\nu \in D'\) and \(\eta < o^K(\nu)\) define a \(K\)-ultrafilter \(W_{\nu, \eta}\) by:
\[
(36) \quad z \in W_{\nu, \eta} \text{ iff } z \in \varphi^K(\nu) \text{ and } \{\alpha | z_\alpha \in (E^\nu_\alpha)_{\eta}\} \in U.
\]

Let \(G^\nu := (W_{\nu, \eta})_{\eta < o^K(\nu)}\). \(G^\nu\) is a sequence of ultrafilters, though at the moment it is not clear that \(G^\nu\) is an extender. Note by the previous section, \(K\) has total extenders on \(\omega_3^V\), so \(o^K(\nu) < \omega_3\) for each \(\nu < \omega_3\) (since we are assuming 0-pistol does not exist; i.e. there are no overlapping extenders).

Consider any \(\nu \in D'\) and any \(X \in S^\nu\) such that there is no truncation at stage \(\theta^\nu_{\alpha^*}\) (recall this holds for \(U\)-many \(\alpha\)). Let \(\pi_X : H_X \rightarrow H_\nu\) be the inverse of the Mostowski collapsing map of \(X\). Since \(E^\nu_\alpha\) is an extender on the mouse \(N^\nu_\alpha\) and \(N^\nu_\alpha\), \(K_X\) agree below their common successor of \(cr(E^\nu_\alpha)\), then \(E^\nu_{\alpha X}\) is \(K_X\)-correct but is not on the \(K_X\) sequence; this implies:
\[
(37) \quad E^\nu_{\alpha X} \text{ is not an element of } H_X.
\]

For each \(\nu \in D'\) define a function \(s^\nu\) by:
\[
(38) \quad s^\nu(\alpha) := \text{the least } \zeta < o^K_\alpha(h_\nu(\alpha)) \text{ such that } G^\nu_{\alpha}(\zeta) \neq (E^\nu_{\alpha})_{\zeta}.
\]

By (37), \(s^\nu(\alpha)\) exists for \(U\)-many \(\alpha\) and \(s^\nu(\alpha) < h_\nu(\alpha)\) (for each \(\nu \in D'\)). Since each ultrafilter \((E^\nu_{\alpha})_{\xi}\) for \(\xi < h_\nu(\alpha) = cr(E^\nu_{\alpha})\) is simply the principal ultrafilter generated by \(\xi\), it is easy to see that \(G^\nu_{\alpha}(\xi) = (E^\nu_{\alpha})_{\xi}\) for each such \(\xi\). So \(s^\nu(\alpha) \geq h_\nu(\alpha)\). To summarize (note we have not yet used the assumption \(|\omega_2 \omega_1/U| = \omega_2\):
\[
(39) \quad h_\nu \leq_U s^\nu < U h_\nu(\kappa(\nu)) \text{ for every } \nu \in D'.
\]
Claim 30. For all but nonstationarily many \( \nu \in S^2_\omega \): \( s^\nu \) is not \( U \)-equivalent to any canonical function of the form \( h_\eta \) (for some \( \eta < \omega_3 \)).

Proof. Suppose to the contrary that there is a stationary \( R \subseteq D' \) and for each \( \nu \in R \), there is an \( \eta^\nu \) such that \( s_\nu =_U \eta^\nu \) \((- =_U \eta^\nu \) (\(- =_U \eta^\nu \)). By (39), we know \( \eta^\nu \in [\nu, \alpha^K(\nu)] \).

Consider the system \( S := \langle (\mu^\nu_-, s^\nu(-) | \nu \in R) \rangle \) of objects from the winning mouse. By Lemma 24 there is a stationary \( R'' \subseteq R \) on which \( S \) lines up. Fix a \( \nu \in R'' \cap \text{Lim}(R'') \). Then:

\[
W_{\nu, \eta^\nu} \text{ is generated by } \langle \eta^\nu' | \nu' \in R'' \cap \nu \rangle \text{ (i.e. } z \in W_{\nu, \eta^\nu} \text{ iff } z \text{ contains a tail of that sequence).} \tag{40}
\]

The proof of (40) is similar to the proof of Claim 27.1. Consider any sequence \( \langle \nu_\alpha | \delta < \omega_2 \rangle \) of points in \( R'' \cap \nu \) which is cofinal in \( \nu \). Pick any \( z \in \varphi^K(\nu) \). For each \( \alpha \) pick a \( \delta(\alpha) \) so that a thread to \( (z)_\alpha \) appears by stage \( \delta(\alpha) \) (in the \( K \) vs. \( K'_\omega \) coiteration). By weak normality of \( U \) there is a \( \delta^* < \omega_2 \) and \( U \)-many \( \alpha \) such that \( \delta(\alpha) \leq \delta^* \). Then show that \( \eta^\nu \in \zeta \) for every \( \delta \in [\delta^*, \omega_2) \); the proof is almost identical to the proof of Claim 27.1. The only difference is that here we use that \( x \) is an element of \( (E^\nu_\alpha)_{h^\nu^\alpha} \) if \( h^\nu^\alpha(\alpha) \in \pi^\nu_{\nu^\delta}^\nu(\alpha) \).

Since \( \nu \) and all the \( \nu_\delta \) are elements of \( R'' \) and \( S \) lines up on \( R'' \), then for each \( \delta < \omega_2 \) there is a set \( A_{\nu_\delta} \subseteq U \) such that \( \pi^\nu_{\nu^\delta}(\mu^\delta_{\alpha_\nu}, \eta^\nu_\nu) = (\mu^\delta_{\alpha_\nu}, \eta^\nu_\nu) \) for every \( \alpha \in A_{\nu_\delta} \). Also by assumption there is a set \( B_{\nu_\delta} \subseteq U \) so that \( h^\nu_{\nu^\delta}(\alpha) = s^\delta(\alpha) \) and \( h^\nu_{\nu^\delta}(\alpha) = s^\nu(\alpha) \) for every \( \alpha \in B_{\nu_\delta} \). Let \( C_\delta := A_{\nu_\delta} \cap B_{\nu_\delta} \). By Lemma 5 there are \( U \)-many \( \alpha \) such that \( I_\alpha = \{ \delta < \alpha | \alpha \in C_\delta \} \) is cofinal in \( \alpha \); let \( C \subseteq U \) be the collection of such \( \alpha \). Then for every \( \alpha \in C \): \( P_\alpha := \langle h^\nu_{\nu^\delta}(\alpha) | \delta \in I_\alpha \rangle \) is a generating sequence for the ultrafilter \( (E^\nu_\alpha)_{h^\nu_{\nu^\delta}(\alpha)} \).

Now pick any \( X \triangleleft (H_\theta , \in , \Delta , ... \) such that \( t_\nu \subseteq X \) and \( \alpha = \text{Cof}(X, \omega_2) \) is an element of \( C \); this is possible because \( C \subseteq U \) and is thus stationary. By (40) and elementarity of \( \pi_X \), \( Q_X := \pi^{-1}_X(\langle \eta^\nu_{\nu^\delta} | \delta < \omega_2 \rangle) = \langle h^\nu_{\nu^\delta}(\alpha_X) | \delta < \alpha_X \rangle \) and this sequence generates the ultrafilter \( \pi^{-1}_X(W_{t_\nu, \eta^\nu}) = \langle \pi^{-1}_X(G^\nu) | h^\nu_{\nu^\delta}(\alpha_X) \rangle \).

But \( P_\alpha \) is a cofinal subset of \( Q_X \). Since \( P_\alpha \) generates \( (E^\nu_\alpha)_{h^\nu_{\nu^\delta}(\alpha_X)} \) and \( Q_X \) generates \( \langle \pi^{-1}_X(G^\nu), h^\nu_{\nu^\delta}(\alpha_X) \rangle \), then these must be the same ultrafilter. But \( h^\nu_{\nu^\delta}(\alpha_X) = s^\nu(\alpha_X) \) (since \( \alpha_X \in C \)). This contradicts the definition of the function \( s^\nu \) in (38).

\[ \square \]

Claim 30, along with (39), imply that many of the mice in the coiterations of this paper have extenders with at least two generators (which is a bit stronger than \( o(\kappa) = \kappa^{++} \)). To see this, pick any \( X \triangleleft (H_\theta , \in , \Delta , ... \) such that \( h^\nu(\alpha) < s^\nu(\alpha) \), where \( \alpha = \text{Cof}(X, \omega_2) \). So in
particular \((E^\nu_\alpha)_{h_\nu(\alpha)}\) is an element of \(H_X\) and if \(E^\nu_\alpha\) had no other generator, then the entire extender \(E^\nu_\alpha\) would be an element of \(H_X\), which contradicts (37).

We also point out the following fact: let \(\tau^\nu < \omega_3\) be minimal such that \(s^\nu \leq_U h_{\tau^\nu}\); by Claim 30, this inequality is in fact strict. Since \(s^\nu >_U h_\xi\) for every \(\xi < \tau^\nu\) and \(s^\nu <_U h_{\tau^\nu}\), then by Lemma 9 the cofinality of \(\tau^\nu\) must be strictly less than \(\omega_2\).

References


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NONREGULAR ULTRAFILTERS ON $\omega_2$

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