COVERING THEOREMS FOR THE CORE MODEL, AND AN APPLICATION TO STATIONARY SET REFLECTION

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ABSTRACT. We prove covering theorems for \( K \), where \( K \) is the core model below the sharp for a strong cardinal, and give an application to stationary set reflection.

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1. Introduction

Jensen’s covering lemma for $L$ expressed a dichotomy: either $0^\#$ exists, or every uncountable set $b$ of ordinals in $V$ is covered by a set in $L$ of the same cardinality. This had the immediate consequence that, e.g., the failure of the Singular Cardinals Hypothesis is a large cardinal property. This covering lemma cannot hold for higher core models: Prikry forcing shows that there cannot be a canonical inner model $K$ which is resistant to set-sized forcing, has a measurable cardinal, and covers every uncountable set in its forcing extensions.

Mitchell [7] proved a kind of covering theorem for the core model below $o(\kappa) = \kappa^{++}$, and obtained a dichotomy similar to Jensen’s: either $K$ recognizes the singularity of singular $V$-cardinals, or else $K$ has large cardinals.

This paper generalizes Mitchell’s result to the following:

**Theorem 1.** (Main Theorem:) Let $K$ be the core model below the sharp for a strong cardinal (i.e., there is no mouse with an overlapping extender). Assume $\omega_2 < \gamma$, $\gamma$ is regular in $K$, and $cf^V(\gamma) < |\gamma|^V$. Then $\gamma$ is measurable in $K$.

Moreover, if $cf^V(\gamma)$ is uncountable then $o^K(\gamma) \geq cf^V(\gamma)$ and for every $\beta < cf^V(\gamma)$, $\{\lambda < \gamma | o^K(\lambda) > \beta\}$ contains a closed unbounded set.

Theorem 1 generalizes Mitchell’s result in several directions: it applies to a larger core model, and does not require $\gamma$ to be a cardinal in $V$. Furthermore, in the case where $cf^V(\gamma)$ is uncountable, Theorem 1 removes the assumption that $(cf^V(\gamma))^\omega = cf^V(\gamma)$ from the main theorem of [7], answering a question from that paper. Vickers [9] had obtained results similar to the “uncountable cofinality” case of Theorem 1 at a smaller core model.

Theorem 1 and its proof, are especially useful in obtaining lower bounds for the consistency strength of combinatorial properties at small regular cardinals (like $\omega_2$ and $\omega_3$). This article gives one such application to stationary set reflection at $\omega_3$:

**Theorem 2.** Let $S^m_n$ denote $\omega_n \cap cof(\omega_m)$. Let $<^*$ be the relation on $\omega_3^V$ defined by: $f <^* g$ iff there is a $< \omega_2$-closed unbounded $C \subset \omega_3$ such that $f(\alpha) < g(\alpha)$ for every $\alpha \in C$. Let $h_\nu$ denote the $\nu$-th canonical

\[1\] Vickers proved the following, regarding ordinals of uncountable cofinality: assume 0-sword does not exist (i.e. there is no mouse with a measure of order > 0) and let $K$ be the core model. If $\omega_2 < \gamma$, $\gamma$ has uncountable cofinality, $cf^V(\gamma) < |\gamma|^V$, and $\gamma$ is regular in $K$, then $\gamma$ is measurable in $K$. If $\omega_2 \leq cf^V(\gamma) < |\gamma|^V$ then $\gamma$ is singular in $K$. 

function on $\omega_3$. Let $o^K$ be the function defined by sending $\lambda$ to its Mitchell order in $K$.

(1) Assume every stationary subset of $S^3_0$ reflects at a point in $\text{cof}(\omega_1)$. Then $h_\nu <^* o^K$ for every $\nu < \omega^+_{3}$. 

(2) (Simultaneous reflection) Assume every pair of stationary subsets of $S^3_0$ have a common reflection point in $\text{cof}(\omega_1)$. Then $o^K(\omega_3) \geq \omega^+_{3}$. 

Section 2 provides some background. Section 3 states and proves a general version of the so-called “Frequent Extensions of Embeddings Lemma.” Section 4 includes some consequences of the Frequent Extensions of Embeddings Lemma, due essentially to Mitchell. Section 5 proves Theorem 1 (and the related Theorem 47); the case where $\text{cf}(\gamma) = \omega$ is the hardest and takes up most of the section. Section 6 proves Theorem 2.
2. Preliminaries

An uncountable set \( X \) will be called \textit{weakly internally approachable} iff there is a strictly \( \subset \)-increasing sequence \( \langle Y_i \mid i < \mu \rangle \) such that \( cf(\mu) > \omega \), \( X = \bigcup_i Y_i \), and for every \( i < \mu \): \( Y_i \in X \), \( Y_i \subset X \), and \( |Y_i| < |X| \). If \( \kappa \geq \omega_2 \) is regular, the collection of weakly internally approachable \( X \) is stationary in \( P_\kappa(H_\theta) \) (in fact closed under sufficiently long \( \subset \) chains). If \( S \subset P_\kappa(H_\theta) \) is stationary, the phrase “property \( P \) holds for almost every \( X \in S \)” means that \( \{ X \in S \mid P \) does not hold for \( X \} \) is nonstationary. The Fodor Lemma for \( P_\kappa(H_\theta) \) states that if \( S \) is stationary and \( F \) is a function on \( S \) such that \( F(X) \in X \) for every \( X \in S \), then there is a stationary \( S' \subset S \) and a set \( Z \) such that for every \( X \in S' \), \( F(X) = Z \).

If \( A \) is a set of cardinality \( \kappa \), a \textit{filtration of} \( A \) is a continuous \( \subset \)-increasing sequence \( \langle A_\alpha \mid \alpha < \kappa \rangle \) such that \( |A_\alpha| < \kappa \) for every \( \alpha < \kappa \) and \( A = \bigcup_{\alpha < \kappa} A_\alpha \). If \( \kappa \) is regular and uncountable then any two filtrations of \( A \) agree on a club. So if \( \nu < \kappa^+ \) and \( \langle A^\nu_\alpha \mid \alpha < \kappa \rangle \) and \( \langle B^\nu_\alpha \mid \alpha < \kappa \rangle \) are filtrations of \( \nu \), then the functions \( \alpha \mapsto otp(A^\nu_\alpha) \) and \( \alpha \mapsto otp(B^\nu_\alpha) \) agree on a club. The equivalence class\(^2\) of \( \alpha \mapsto otp(A^\nu_\alpha) \) is called the \( \nu \)-th canonical function on \( \kappa \); in this paper \( h_\nu \) will denote a representative of this class. There are also inductive definitions of \( h_\nu \); see e.g. \cite{3}.

2.1. Fine structure and ultrapowers. All fine-structural notions used in this paper can be found in \cite{10} (e.g. fine-structural ultrapowers). We fix some notation and state a useful lemma.

**Definition 3.** If \( M \) is an acceptable \( J \)-structure, \( \phi \) is an \( \Sigma_0^n \) formula, \( \tau \) is an \( M \)-cardinal with \( \tau \leq \omega_1^M \), and \( f_1, \ldots, f_k \) are good \( \Sigma_1^{n-1}(M) \) functions with \( \gamma_i := \text{dom}(f_i) \in \tau \) for each \( i \), then \( w^M_{f_1, \ldots, f_k} := \{ \langle \xi_1, \ldots, \xi_k \rangle \mid \langle \xi_1, \ldots, \xi_k \rangle \in \prod_{i \leq k} \gamma_i \text{ and } M \models \phi(f_1(\xi_1), \ldots, f_k(\xi_k)) \} \); note this is a bounded subset of \( \tau \). The symbols \( u^M_{f,g} \) and \( u^M_{f,g} \) will stand for the sets \( \{ \xi \in \text{dom}(f) \cap \text{dom}(g) \mid f(\xi) = g(\xi) \} \) and \( \{ \xi \in \text{dom}(f) \cap \text{dom}(g) \mid f(\xi) \notin g(\xi) \} \) respectively.

**Lemma 4.** (Interpolation Lemma) Assume \( \sigma : M \rightarrow \Sigma_0^{n+1} M^* \) where \( M \) end extends \( Q = J^A, \tau \) is a cardinal in \( M \), and \( \omega_1^M \geq \tau \). Let \( \tau := \sup(\sigma[\tau]) \), and \( Q' := M|\tau \). Then \( M' := \text{ult}^n(M, \sigma \upharpoonright \tau) \) is wellfounded. Let \( \tilde{\sigma} : M \rightarrow M' \) be the \( n \)-ultrafilter map. There is also a unique \( \Sigma_1^{n-1} \)-preserving map (if \( n = 0 \) this map is generally \( \Sigma_0 \) preserving) \( \tilde{\sigma}' : M' \rightarrow M^* \) such that \( \sigma' \upharpoonright \tau' = \text{id} \) and \( \sigma' \circ \tilde{\sigma} = \sigma \). If \( \tilde{\sigma} \) is \( n \)-cofinal (e.g., if \( R^n_M \neq \emptyset \), or if \( n = k(M, \tau) \)) then \( \sigma' \) is \( \Sigma_0^{(n+1)} \) preserving.

\(^2\)modulo the equivalence relation on \( \Sigma^V \) defined by “agree on a club.”
We state the following basic fact about cutpoints, which will be used in section 5:

**Fact 5.** Let \( \pi : M \to F N \) and \( \gamma < \pi(\kappa) \); without loss of generality, assume \( F \) is whole. Then \( \gamma \) is a cutpoint of \( F \) if and only if \( \pi(\kappa) \cap h_N[\text{range}(\pi) \cup \gamma] = \gamma \) (i.e., \( \gamma \) is not a cutpoint iff \( h_N[\text{range}(\pi) \cup \gamma] \cap [\gamma, \pi(\kappa)) \neq \emptyset \)).

2.2. Premice with non-overlapping extenders. This section summarizes some facts about premice for non-overlapping extenders. The definition of a bookkeeping premouse, which appears on page 252 of \([10]\), is rather technical, and the top predicate of a bookkeeping premouse is not an amenable predicate (so the fine structure theory does not apply to it). If \( M \) is a bookkeeping premouse then the expansion of \( M \), denoted by \( \hat{M} \) is an acceptable \( J \)-structure and is in fact a coherent structure; again see page 252 of \([10]\) for the definition. A bookkeeping premouse and its expansion can be computed from each other. In coiterations between mice, the notion of “least disagreement” is unambiguous if we’re referring to bookkeeping premice, but is somewhat ambiguous if we’re referring to their expansions; this is why bookkeeping premice are used.

We use the following notations related to the Mitchell order (these agree with \([10]\)):

**Definition 6.** For \( \kappa \in M \), \( O^M(\kappa) := \{ \nu \geq \kappa^+ M | E^M_\nu \neq \emptyset \} \), \( o^M_*(\kappa) = \text{otp}(O^M(\kappa)) \), and \( o^M(\kappa) \) is the least primitive recursively closed \( \nu \geq \kappa^+ M \) which does not index an extender with critical point \( \kappa \). \( o^M(\kappa) \) is called the order of \( \kappa \) in \( M \), and \( o^*_M(\kappa) \) is called the Mitchell order of \( \kappa \) in \( M \). Note that by the coherency requirement of premice, \( o^*_M(\kappa) \geq \beta \) implies that the Mitchell order (in the usual sense) of \( \kappa \) is at least \( \beta \). If \( F \) is an extender over \( M \) with critical point \( \kappa \) and \( \text{ult}(M,F) \) is wellfounded, we define \( o(F) \) as \( o_{\text{ult}(M,F)}(\kappa) \), and similarly for \( o_*(F) \). If \( F = E^M_\nu \) is an extender on the sequence of the premouse, then \( O^M(F) := \{ \eta \in [\tau, \nu] | E_\eta \neq \emptyset \} \) and \( o^*_M(F) \) is defined as the order-type of \( O^M(F) \); i.e. \( O^M(F) \) is \( O_{\text{ult}(\hat{M},F)}(\kappa) \) and \( o^*_M(F) \) is \( o_{\text{ult}(M,F)}(\kappa) \).

Here are a few basic facts about these premice which appear in section 8.1 of \([10]\):

**Lemma 7.**

1. If \( \kappa^+ M \leq \gamma < o^M(\kappa) \) and \( \gamma \) is primitive recursively closed then \( cr(E_\gamma) = \kappa \);
2. If there is a (possibly partial) extender on \( M \)’s extender sequence with critical point \( \kappa \), then \( o^M(\mu) < \kappa \) for every \( \mu < \kappa \).
(3) If \( M = (J^E_\alpha, E_\alpha) \) is a bookkeeping premouse and \( \text{cr}(E_\alpha) = \kappa \),
then \( \alpha = \sigma^M(\kappa) \) and \( \pi(\kappa) > \alpha \), where \( \pi : J^E_\tau \to E_\alpha \).
\( J^E_\tau \) and \( \tau = \kappa^{+M} \).

(4) If \( F \) is on the extender sequence of a premouse \( M \), then \( F \) has no cutpoints.\(^3\)

(5) The property of being a premouse without a top extender is a \( Q \)-property.

(6) Premousehood is a \( \Pi_2 \) property for transitive rudimentarily closed
structures. \( M = (J^E_\alpha, \emptyset) \) is a premouse (i.e. the expansion of a
bookkeeping premouse iff:
(a) \( (J^E_\alpha, \emptyset) \) is a premouse;
(b) \( M \) is a coherent structure;
(c) \( F \) is weakly amenable with respect to \( J^E_\alpha \). For coherent
structures with top predicate \( F \), this is equivalent to \( F \) measuring all subsets of \( \kappa \) in the coherent structure;
(d) All generators of \( F \) are \( < \alpha \).

(7) If \( G \) is a weakly amenable extender with respect to \( \hat{M} \) and
\( \text{cr}(G) \leq \text{height}(M) \), then \( \text{ult}(\hat{M}, G) \) is a premouse (assuming
it is wellfounded).

(8) Assume \( Q \) is an initial segment of \( M \), \( \sigma : Q \to \Sigma_0 \) \( Q' \) is cofinal,
and \( \text{ult}^n(\hat{M}, \sigma) \) is wellfounded; let \( \bar{\sigma} : \hat{M} \to \text{ult}^n(\hat{M}, \sigma) \). If
either:
- \( n > 0 \), or
- \( n = 0 \) and \( \text{cr}(E^{\text{top}}_{\text{eq}})^{+\hat{M}} \geq \text{ht}(Q) \)
then \( \text{ult}^n(\hat{M}, \sigma) \) is a premouse.

Let \( M = (J^E_\alpha, E_\alpha) \) be a bookkeeping premouse with \( E_\nu \neq \emptyset, \kappa = \text{cr}(E_\nu) \), and \( \tau = \kappa^{+M} \). Suppose \( \beta > \kappa \) is a generator for \( E_\nu \). Then \( \beta \)
must be strictly between \( \tau \) and \( \nu \), and must be a fixed point for the
enumeration of p.r.c. ordinals \( \geq \tau \). In other words, to be a generator
\( \beta \) would have to be the \( \beta \)-th p.r.c. ordinal \( \geq \tau \). Consequently, if \( o^M_\nu(E_\nu) \leq \kappa^{+M} \) (or even a bit higher) then \( E_\nu \) has only one generator,
namely, its critical point. Such an extender is essentially a measure, as
the ultrapower can be constructed using only \( E_{\nu, \kappa} \). For this reason, we
have the following convention for the rest of the article:

**Convention 8.** If \( o^M_\nu(E_\nu) = \gamma \) is sufficiently small, then we will usually
view \( E_\nu \) as simply the measure \( E_{\nu, \kappa} \), and denote it by \( U^M(\kappa, \gamma) \).\(^4\)

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\(^3\)A cutpoint on a coherent structure like \( \hat{M} \) would yield a superstrong extender
in \( \hat{M} \), which is far beyond the core model used in this paper.

\(^4\)See Definition 6 for the meaning of \( o^M \), \( o^M_* \), and \( O^M \).
Iterations of premice and the associated concepts (iteration indices, truncations, etc.) are defined in the usual way, with the slight technical difference that the domains of the iteration maps are expansions of bookkeeping premice. All iterations of premice in this paper are linear; i.e. iteration trees are not necessary here. Unless stated otherwise, iteration means that *-ultrapowers are always used. An iteration is simple if there are no truncations in the iteration. Since the extender $E_\nu$ is close to $\hat{M}|\nu$ for any premouse $M$, total iteration maps are $\Sigma^*$-preserving. For $k \in \omega$, a $k$-iteration of a premouse $M$ is an iteration that uses $k$-ultrapowers until the first truncation, and then *-ultrapowers at all later stages. A degenerate $(k-) iteration of M is a concatenation of $\omega$-many simple $(k-) iterations which starts with M and has a truncation at each concatenation point. A premouse $M$ is called $(k-) iterable or a mouse if and only if every $(k-) iteration of M can be continued, and there are no degenerate $(k-) iterations of M. An iteration is normal iff the iteration indices are strictly increasing, and $M$ is called normally $(k-) iterable iff every normal $(k-) iteration of $M can be continued. An iteration is above $\kappa$ iff all the critical points of the iteration are above $\kappa$. $M$ is (normally) iterable above $\kappa$ iff every (normal) iteration of $M$ above $\kappa$ can be continued.

Coiterations (i.e. comparisons) of premice are defined as usual, and for mice with nonoverlapping extenders the coiteration process is linear. If $M$ and $N$ are coiterable then the length of the coiteration is $< \max |M|^+, |N|^+$. Furthermore, if every proper initial segment of $M$ and $N$ are solid (this is a $\Pi_1$ condition called presolidity), then at least one side of the coiteration is simple. We will see below that iterability guarantees this condition, and more.

We fix some notation regarding coiterations; for simplicity this definition will only apply to coiterations where one side is simple:

**Definition 9.** Let $M, N$ be coiterable premice, and suppose their coiteration has length $\theta+1$. Assume the $M$ side of the coiteration is simple. The coiteration data is

$$\langle M_i, \pi^M_{i,j}, N_i, \pi^N_{i,j}, \nu_i, \kappa_i, \tau_i, \delta_i, (N_i)^*|i \leq j \leq \theta \rangle$$

where:

- For $i < \theta$:
  - $M_i, \pi^M_{i,j}$ are the iterates and iteration maps on the $M$ side, and similarly for the $N$ side (if there is a truncation between stages $i$ and $j$ then $\pi^N_{i,j}$ is only partial).
  - $\nu_i$ is the least point of disagreement between $N_i$ and $M_i$. 

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- $\kappa_i$ is the corresponding critical point of whichever side has an extender indexed at $\nu_i$ (if $\nu_i$ indexes an extender in both $M_i$ and $N_i$ then $\text{cr}(E_{\nu_i}^{M_i}) = \text{cr}(E_{\nu_i}^{N_i})$).
- $\tau_i = \kappa_i^{+M_i}$
- $\delta_i$ is maximal such that $\tau_i$ is a cardinal in $\hat{N}_i|\delta_i$; note that if $\delta_i < \text{ht}(N_i)$ then the ultimate projection of $(N_i)^*$ is $\leq \kappa_i$ and $(N_i)^*$ is fully sound.

- For $i = \theta$: if $M_\theta$ is a proper initial segment of $N_\theta|\kappa_\theta + N_\theta$ for some $\kappa$, then we define $\kappa_\theta := \kappa$, $\tau_\theta = \kappa^{+M_\theta}$, $\delta_\theta$ to be maximal so that $\tau_\theta$ is a cardinal in $N_\theta|\delta_\theta$, and $(N_\theta)^* := N_\theta|\delta_\theta$. Note that such a $\kappa_\theta$ is not literally a critical point of the coiteration.

The following is often called the copying construction; we state the version for $k-$iterations:

**Definition 10.** Assume $\hat{M}, M$ are premice, $\sigma : \hat{M} \rightarrow \Sigma_0^{(k)} \hat{M}$, and $\overline{I}$ is a $k-$iteration of $\hat{M}$. The $k-$copy of $\overline{I}$ via $\sigma$ consists of the following (which may be vacuous; the next lemma gives sufficient conditions for this copying construction to be carried out):

- A $k-$iteration $I$ of $M$ of the same length as $\overline{I}$; denote this length by $\theta$ and the iteration maps by $\pi_{i,j}$;
- For each $i < \theta$ a map $\sigma_i : \hat{M}_i \rightarrow \Sigma_0^{(k)} \hat{M}_i$ where $\sigma_0 = \sigma$, $\pi_{i,j} \circ \sigma_i = \sigma_j \circ \pi_{i,j}$ and $\sigma_j \upharpoonright \nu_i = \sigma_i \upharpoonright \nu_i$ for every $i \leq j < \theta$. The maps $\sigma_i$ are called copies of $\sigma$.
- $\sigma_i(\nu_i) = \nu_i$ for every $i < \theta$.

Note that if $\overline{I}$ is normal and its $k-$copy to $M$ exists, then the copy is also normal. Also, the truncation stages of the copy are the same as the truncation stages of $\overline{I}$. The following is a sufficient condition for the copying construction to be carried out; again we state the version for $k-$iterations:

**Lemma 11.** Assume $\sigma : \hat{M} \rightarrow \Sigma_0^{(k)} \hat{M}$, $R_{\overline{I}}^{k_M} \neq \emptyset$ if $k > 0$, and $M$ is $k$-iterable. Then the $k-$copying construction can be carried out.

**Proof.** The proof is similar to Lemma 4.3.1 in [10]. In the current version, the assumption that $R_{\overline{I}}^{k_M} \neq \emptyset$ (if $k > 0$) is used to ensure that the ultrapower maps in the iterations and the copy maps remain $\Sigma_0^{(k)}$ preserving (even if the degree of the critical points of the coiteration are strictly less than $k$).

The copying construction (for *-iterations) is used to prove the Dodd-Jensen Lemma, which states that if $M$ is iterable, $\overline{I}$ is an iteration of
M resulting in $M'$, and there is a $\Sigma^*$-preserving map $\sigma : \hat{M} \to \hat{M}'$, then $I$ is simple and the iteration map from $M \to M'$ is pointwise smaller than $\sigma$. One consequence of this is that a given mouse $Q$ cannot be both a simple and a non-simple iterate of $M$.

There is also a version of the Dodd-Jensen Lemma for $k$-iterable mice which have $k$-very good parameters, and uses the copying construction in Definition 10:

**Lemma 12.** Assume $M$ is $k$-iterable, $R_k^\hat{M} \neq \emptyset$ if $k > 0$, there is some $\sigma : \hat{M} \to \Sigma_0^{(k)} \hat{M}'$, and $M'$ is a $k$-iterate of $M$. Then there is only one $k$-iteration $I$ from $M \to M'$, this $k$-iteration is simple, and the corresponding map is pointwise $\leq \sigma$.

The Solidity Theorem states that every mouse is solid. This has several important consequences, including the Condensation Lemma and the fact that at least one side of a coiteration of mice is simple. The latter leads to the canonical wellordering of mice, where $M <^* N$ means that the $N$ side truncates in the coiteration of $M$ vs. $N$; this is equivalent to the statement that there is some mouse $Q$ which is a nonsimple iterate of $N$ and a simple iterate of $M$. The following is an often-used consequence of solidity (and the Dodd-Jensen lemma):

**Lemma 13.** Assume $M$ is iterable and $\sigma : \hat{M} \rightarrow_{\Sigma_0^{(n)}} \hat{M}$, and that in the “attempted” coiteration of $\hat{M}$ with $\hat{M}$, the $\hat{M}$ side is above $\omega^{\rho_{\hat{M}}^{n+1}}$; i.e. $\hat{M}$ and $M$ agree below $\omega^{\rho_{\hat{M}}^{n+1}}$. Then $\hat{M}$ and $M$ are coiterable, the $\hat{M}$ side is simple, and $\hat{M}$ is iterable.

We also state a version for $k$-iterations:

**Lemma 14.** Assume $M$ is $k$-iterable, $\sigma : \hat{M} \rightarrow_{\Sigma_0^{(k)}} \hat{M}$, and $R_k^\hat{M}$ and $R_k^\hat{M}$ are both nonempty if $k > 0$. Then $\hat{M}$ and $M$ are $k$-coiterable, the $\hat{M}$ side is simple, and $\hat{M}$ is $k$-iterable.

**Proof.** Since $M$ is $k$-iterable then every proper initial segment is $^*$-iterable; so the usual Solidity Theorem implies $\hat{M}$ is presolid. Presolidity is a $\Pi_1$ property, and so $\hat{M}$ is presolid as well. Furthermore, the preservation degree of $\sigma$ and $R_k^\hat{M} \neq \emptyset$ (for $k > 0$) implies by Lemma 11 that $\hat{M}$ is $k$-iterable. Let $Q$ be the last premouse in the $k$-coiteration of $\hat{M}$ with $M$. Since both $\hat{M}, M$ are presolid then at least one side of the $k$-coiteration is simple. The $\hat{M}$ side must be simple. If not, let $\pi_{M,Q}$ be the $k$-iteration map on the (simple) $M$ side; $R_k^\hat{M} \neq \emptyset$ guarantees that $\pi_{M,Q}$ is $\Sigma_0^{(k)}$ preserving, even if the degrees of the critical points used
in the iteration are strictly less than \( k \). Then \( \pi_{M,Q} \circ \sigma \) would be a \( \Sigma^0_k \) map from \( \bar{M} \to Q \), but \( Q \) is also a non-simple \( k \)-iterate of \( \bar{M} \). This contradicts Lemma [12].

The following lemma appears as Lemma 8.2.2 in [10].

**Lemma 15.** Assume \( \sigma : \hat{M} \to \Sigma_0 \hat{N} \) where \( M, N \) are 0-iterable. If \( \mu < cr(\sigma) \), then \( M, N \) agree below \( o^M(\mu) \). Consequently, if these two premises are coiterable, then the \( M \) side is above \( o^M(\mu) \).

In this paper, lemmas 13/14 and 15 are often used in conjunction; Lemma 15 is used first to show that the conditions of Lemma 13 or 14 are met.

The following are versions of Lemmas 8.2.10 and 8.2.11 in [10]:

**Lemma 16.** Assume \( M \) is a mouse which end-extends \( Q = J^E_\tau \), \( \kappa \) is the largest cardinal in \( Q \), \( \hat{M} \) is sound above \( \kappa \), \( cf(\tau) > \omega \), \( n = \text{degree}_{\hat{M}}(\kappa) \), and \( \sigma : Q \to \Sigma_0 \hat{Q} \) is cofinal. Then \( P := \text{ult}(\hat{M}, \sigma) \) is wellfounded. If \( P \) is a premouse then it is \( n \)-iterable. If \( P \) is not a premouse, then the “premouse associated with \( P \)” (defined below) is \( m \)-iterable, where \( \mu = cr(E_{\text{top}}^P) \) and \( m \) is the degree of \( \mu \) in \( \hat{P} \| \delta \).

Although the statement of lemma 16 in [10] only mentions that \( \hat{M} \) is normally iterable above \( \sigma(\kappa) \), the proof there—in conjunction with Lemma 14—really shows that \( \hat{M} \) is fully \( n \)-iterable. In this paper, Lemma 16 will be used only when protomice arise during the proof of the Frequent Extensions of Embeddings Lemma. In those cases, the “premouse replacements” for the protomice can be viewed as canonical liftings of the sort guaranteed by Lemma 16.

There is also the coarse version of the last lemma:

**Lemma 17.** Assume \( M \) is a 0-iterable premouse which end-extends \( Q = J^E_\tau \), \( cf(\tau) > \omega \), \( Q \) has a largest cardinal \( \kappa \), \( o^M(\mu) < \tau \) for every \( \mu < \kappa \), and \( \sigma : Q \to \Sigma_0 \hat{Q} \) is cofinal. Then the coarse ultrapower \( \text{ult}(\hat{M}, \sigma) \) is wellfounded and 0-iterable.

**Corollary 18.** Assume \( M \) is a mouse which is sound above \( \kappa \), \( F \) is an extender over \( M \) with critical point \( \kappa \), and \( \tau := \kappa^+ \) has uncountable cofinality. If \( \text{ult}(M|\tau, F) \) is wellfounded (recall that \( M|\tau \) denotes \( J^E_\tau \), i.e. \( M \| \tau \) without the top predicate), then \( \text{ult}^*(\hat{M}, F) \) is a wellfounded premouse and is normally iterable above \( \pi(\kappa) \), where \( \pi \) is the *-ultrapower map of \( \hat{M} \) by \( F \).

**Proof.** This is a consequence of Lemma 16; note \( \text{ult}^*(\hat{M}, F) = \text{ult}^*(\hat{M}, \pi) \) where \( \pi : M|\tau \to F \text{ ult}(M|\tau, F) \).
Corollary 19. Assume $M$ is a $0$-iterable bookkeeping premouse, $\kappa$ is a cardinal in $\hat{M}$, $\tau := \kappa + \hat{M}$ has uncountable cofinality, and $o^M(\mu) < \tau$ for every $\mu < \kappa$. Let $F$ be an extender over $M$ with critical point $\kappa$. If $\text{ult}(M|\tau,F)$ is wellfounded, then $\text{ult}(\hat{M},F)$ is a wellfounded $0$-iterable premouse.

Proof. This is a consequence of Lemma 17; note $\text{ult}(\hat{M},F) = \text{ult}(\hat{M},\pi)$ where $\pi : M|\tau \to F \text{ult}(M|\tau,F)$.

The following lemma is a useful condition which guarantees the uniqueness of an extender that can appear on a premouse sequence.

Lemma 20. Assume $M = (J_\alpha^E,F_M)$ and $N = (J_\alpha^E,F_N)$ are $0$-iterable bookkeeping premice, $\kappa := \text{cr}(F_M) = \text{cr}(F_N)$, and there is some $\tau \in (\kappa,\alpha]$ of uncountable cofinality which is a successor cardinal in both $\hat{M},\hat{N}$. Then $M = N$ (i.e. $F_M = F_N$).

Proof.

Claim 21. $\hat{M},\hat{N}$ have the same successor of $\alpha$ and agree below this successor.

We show that $\alpha^{+\hat{M}} \leq \alpha^{+\hat{N}}$ and that they agree below $\alpha^{+\hat{M}}$; a symmetric argument shows the reverse.

Consider the ultrapower map $\pi_N : J_\tau^E \to F_N J_{\tau'}^N$; note that since $\tau$ is a successor cardinal then $J_\tau^E \models ZFC^-$, so $J_{\tau'}^N$ is an initial segment of $\text{ult}(J_\alpha^E,F_N)$, which in turn is the domain of the expansion $\hat{N}$. Since $M$ end-extends $J_\tau^E$ (note we do not require that $\tau$ is the successor of $\kappa$; $\tau$ might be larger than $\kappa + \hat{M}$) and $\alpha = o^{\hat{M}}(\kappa)$ by Lemma 7 then Lemma 17 guarantees the coarse lifting of $\pi_N$ to $\hat{M}$ is wellfounded and $0$-iterable. Let $\pi : \hat{M} \to \hat{M}'$ be this coarse lifting (note it is just the ultrapower map of $\hat{M}$ by $F_N$). So $\hat{M}'$ agrees with $\hat{N}$ below the common cardinal $\pi(\tau)$. In particular, they have the same $\alpha^+$ and agree below it. Furthermore, by coherency of $F_N$ we have $E^M' | \alpha = E | \alpha$ and $E^M_\alpha = \emptyset$. So at the first stage of the $0$-coiteration of $M$ with $M'$, the $M$ side applies $F_M$ and the $M'$ side does nothing. Let $\hat{M}_1 = \text{ult}(\hat{M},F_M)$ denote the first iterate on the $M$ side; note that the ultrapower used to create $M_1$ extends the coherency map $c_M : J_{\kappa + M}^E \to F_M |M|$ and that $c_M(\kappa) > \alpha$. So $M_1$ and $M$ have the same $\alpha^+$ and agree below it.

The remainder of the coiteration is strictly above $\alpha$, and by the coarse version of the Dodd-Jensen Lemma the $M_1$ side is simple. So $\alpha^{+\hat{M}_1} \leq \alpha^{+\hat{M}'}$ and $\hat{M}_1$ agrees with $\hat{M}'$ below $\alpha^{+\hat{M}}$. And $\hat{M}'$ agrees with $\hat{N}$ on the same part.
Claim 22. \( F_M = F_N \).

Let \( \lambda \) be the common cardinal predecessor of \( \tau \) in \( M, N \). For a large regular \( \theta \), fix some countable elementary substructure \( X \prec (H_\theta, \in, M, N, \tau, \lambda, \alpha, \ldots) \) and let \( \sigma : \dot{H} \to H_\theta \) be the inverse of the Mostowski collapsing map. Let \( (\dot{M}, \dot{N}, \dot{\tau}, \dot{\lambda}, \alpha, \ldots) = \sigma^{-1}(M, N, \tau, \lambda, \alpha\ldots) \). We will show that \( \dot{F}_M = \dot{F}_N \).

Let \( \bar{\tau} = \text{sup}(\sigma[\bar{\tau}]) \); this is \( < \tau \) because \( \text{cf}(\tau) \) is uncountable. Using the coarse interpolation lemma, construct interpolates \( \dot{M}, \dot{N} \) and the following maps:

- \( \bar{\sigma}_M : \dot{\widehat{M}} \to \dot{M} \) and \( \bar{\sigma}_N : \dot{\widehat{N}} \to \dot{N} \) which extend \( \sigma \upharpoonright \dot{M} \) and \( \sigma \upharpoonright \dot{N} \), respectively;
- \( \sigma'_M : \dot{\widehat{M}} \to \dot{M} \) and \( \sigma'_N : \dot{\widehat{N}} \to \dot{N} \) such that the critical point of both of these maps is \( \bar{\tau} \).

By Lemma (15) and the fact that \( \sigma'_M \upharpoonright \bar{\tau} = \text{id} \), then \( \dot{M} \) is an initial segment of \( M \). In fact \( \dot{M} \) is a proper initial segment of \( Q := M\upharpoonright \tau \), since \( \bar{\tau} = \lambda^+\dot{M} < \tau = \lambda^+M \). Similarly, \( \dot{N} \) is a proper initial segment of \( Q \).

By claim (21) and the elementarity of \( \sigma, \dot{M} \) and \( \dot{N} \) have the same successor of \( \bar{\alpha} \) and agree below this successor. For the rest of this proof let \( \bar{\alpha}^+ \) denote this common successor, and let \( \dot{E} := E_M \upharpoonright \bar{\alpha}^+ = E_N \upharpoonright \bar{\alpha}^+ \). Now \( \bar{\sigma}_M \upharpoonright \bar{\alpha}^+ = \bar{\sigma}_N \upharpoonright \bar{\alpha}^+ \), since these restrictions are both just the coarse lifting of \( \sigma \upharpoonright \bar{\tau} \) to the domain \( J_{\bar{\alpha}^+}^E \). Let \( \bar{\alpha} \) denote the ordinal \( \bar{\sigma}_M(\bar{\alpha}) = \bar{\sigma}_N(\bar{\alpha}) \). Then \( E_{\bar{\alpha}}^M = E_{\bar{\alpha}}^N = E_{\bar{\alpha}}^Q \), since \( \dot{M} \) and \( \dot{N} \) are both initial segments of \( Q \). This implies that \( \dot{F}_M = \dot{F}_N \), which finishes the proof. \( \square \)

Corollary 23. Assume \( M, N \) are 0-iterable premice, \( \mu \) is an element of both \( M, N \), and there is some \( \tau \) such that \( \mu < \tau \leq \min(o^M(\mu), o^N(\mu)) \), \( \text{cf}(\tau) \) is uncountable, and \( \tau \) is a successor cardinal in both \( M, N \). Then \( M, N \) agree below \( \min(o^M(\mu), o^N(\mu)) \).

Proof. Suppose \( \beta \) indexes an extender on \( \mu \) in both premice and they agree below \( \beta \). Then the pair \( M\upharpoonright\beta \) and \( N\upharpoonright\beta \) satisfies the hypothesis of the previous lemma, so \( E^M_\beta = E^N_\beta \). \( \square \)

Many arguments involve an iteration of limit length where a “critical point is repeated cofinally often” in the iteration and there has been a truncation. The following lemma shows how to identify such points from the last iterate, and is essential to the proofs in section 5.

Lemma 24. Let \( \langle N_i, \pi_{i,j}, \kappa_i, \nu_i | i \leq j < \theta \rangle \) be a normal iteration of a mouse \( N_0 \) such that \( \theta \) is a limit ordinal and there is a truncation at
some stage < θ; let \( \hat{i} \) be the largest truncation stage. Assume there is a cofinal \( I \subset \theta \) such that \( \pi_{i,j}(\kappa_i) = \kappa_j \) for every \( i < j \) which are both in \( I \). Let \( \tilde{\kappa} := \sup\{\kappa_i | i < \theta\} \) and let \( n \) be the degree of \( \tilde{\kappa} \) in \( \hat{N}_\theta \) (i.e. \( n \) is the maximal natural number so that \( \tilde{\kappa} \) is strictly less than the \( n \)-th projectum).

Then \( D := \{\kappa_i | i < \theta \text{ and } \pi_{i,\theta}(\kappa_i) = \tilde{\kappa}\} \) is a closed unbounded subset of \( \tilde{\kappa} \) and for every \( \beta \in [\min(D), \tilde{\kappa}] \), \( \beta \in D \iff \hat{h}[\beta] \text{ does not intersect } [\beta, \tilde{\kappa}] \), where \( \hat{h} := \hat{h}^{n+1}_{\hat{N}_\theta}(-, p_{\hat{N}_\theta}) \).

**Proof.** That \( D \) is a closed and unbounded subset of \( \tilde{\kappa} \) follows from the existence of the cofinal set \( I \) and basic properties of direct limits. The rest of the proof is about the characterization of \( D \) using the fine structure of \( \hat{N}_\theta \).

For the forward direction, assume \( \beta \in D \). Then it is a critical point on the thread to \( \tilde{\kappa} \); say \( \beta = \kappa_i \). So for any \( \xi < \beta \), \( \hat{h}(\xi) = \hat{h}(\pi_{i,\theta}(\xi)) \) is in the range of \( \pi_{i,\theta} \) and thus can’t be in the interval \( [\kappa_i, \tilde{\kappa} = \pi_{i,\theta}(\kappa_i)] \).

Since \( \hat{i} \) is a truncation stage, then for every \( i \in (\hat{i}, \theta) \) the ultimate projectum of \( \hat{N}_i \) is \( \leq \kappa_i \) and \( \hat{N}_i \) is sound above \( \kappa_i \) (and the same is true of the truncated version of \( N_\theta \)). This is a straightforward proof by induction on the stages \( \geq \hat{i} \); see e.g. the proof of Lemma 6.6.4 in [10].

From now on, assume \( \beta \notin D \) and \( \beta > \min(D) \). For such \( \beta \), let \( i \) be the minimal such that \( \beta \in \text{range}(\pi_{i,\theta}) \) and \( i \geq i_0 \), where \( i_0 > \hat{i} \) is the stage such that \( \kappa_{i_0} = \min(D) \). Since we assume \( \beta \geq \min(D) \), then in fact \( i > i_0 \) and is thus a successor stage; in fact, \( i \) is the first stage where a thread to \( \beta \) appears. To see that \( i > i_0 \), just note that \( \beta \in (\kappa_{i_0}, \tilde{\kappa}) \) yet \( \text{range}(\pi_{i_0,\theta}) \) does not intersect \( [\kappa_{i_0}, \tilde{\kappa}] \) (since \( \tilde{\kappa} = \pi_{i_0,\theta}(\kappa_{i_0}) \)). So \( i = j + 1 \) for some stage \( j \). Also, we may WLOG assume the degrees of the critical points past stage \( i_0 \) are fixed at \( n \) (in fact all points on the thread \( D \) are strictly between the \( (n+1) \)-st and \( n \)-th projecta) and the iteration maps restricted to the \( n \)-th reduces agree with the coarse ultrapower on those reducts. Let \( \beta_i = (\pi)_{i,\theta}^{-1}(\beta) \). If \( \beta_i < \pi_{j,j+1}(\kappa_j) \), then since \( \beta_i \) is not a cutpoint of \( E_{\nu_j} \) (mice below 0-pistol do not have cutpoints; see Lemma 7), Fact 5 implies that there is some \( \xi < \beta_i \) and \( f \in H^0_{N_j} \) such that \( \pi_{j,j+1}(f)(\xi) \in [\beta_i, \pi_{j,j+1}(\kappa_j)] \). Since \( \hat{N}_j \) is sound above \( \kappa_j \) then \( \pi_{j,j+1}(f)(\xi) \) is \( \Sigma_4^{(n)}(\hat{N}_{j+1}) \) definable using a parameter \( < \kappa_j \) and \( p_{\hat{N}_{j+1}} \). Applying \( \pi_{j+1,\theta} \) to \( \pi_{j,j+1}(f)(\xi) \) yields an element of \( \hat{h}[\beta] \cap [\beta, \tilde{\kappa}] \) and finishes the case.

\footnote{We do not assume \( \tilde{\kappa} \) has uncountable cofinality.}
Similarly, if $\beta_i \geq \pi_{j+1}(\kappa_j)$ (or simply above the strict supremum $s_j$ of $E_{\nu_j}$’s generators) then there is some $\xi < s_j \leq \beta_i$ and good $\Sigma_1^{\infty}(\hat{N}_j)$ function $f$ in parameter $r$ such that $\beta_i = \pi_{j+1}(f)(\xi)$. Again, since $\hat{N}_j$ is sound above $\kappa_j$, $\pi_{j+1}(f)(\xi)$ is $\Sigma_1^{\infty}(\hat{N}_j)$ definable from $\xi, p_{j+1}$, and some parameter $< \kappa_j$. Then apply $\pi_{j+1,\theta}$ to $\pi_{j+1}(f)(\xi)$ to yield an element of $\hat{h}[\beta] \cap [\beta, \theta]$. 

2.3. Protomice. Suppose $Q$ is an initial segment of a premouse $M$, and $\sigma : Q \to Q'$ is $\Sigma_0$ and cofinal. Assume $\omega \rho^\infty_{\hat{M}} \geq \gamma := \text{ht}(Q)$, and that $\hat{\sigma} : \hat{M} \to P$ is the canonical $n$-lifting of $\sigma$ to $\hat{M}$. If $n > 0$ then $\hat{\sigma}$ is $\Sigma_2^{\infty}$ preserving; since premouseness is a $\Pi_2$ property, this implies that $P$ is a premouse (i.e. the expansion of a bookkeeping premouse).

Now assume $n = 0$ and $\hat{F}$ is the top extender of $\hat{M}$ with critical point $\mu$. If $\text{ht}(Q)$ is a successor cardinal in $\hat{M}$ with cardinal predecessor $\kappa$, then the coarse lifting $\hat{\sigma}$ is continuous on $\hat{M}$-cofinalities $> \kappa$; so if $\mu \geq \kappa$ then $\mu + \hat{M}$ is mapped continuously by $\hat{\sigma}$, and this implies that again $P$ is a premouse. Similarly, if $\gamma$ is limit and $\mu \geq \gamma$ then $P$ is a premouse.

The remaining case is when $n = 0$ and $\mu + \hat{M} < \gamma$. Since $\hat{\sigma}$ is $\Sigma_0$ and cofinal, $P$ is still a coherent structure, but its top extender $\hat{F}$ may not be weakly amenable. If this happens, $P$ is called a protomouse. Let $c_P : |P|^{\hat{\sigma}} \to \hat{F}^{|P|}$ witness that $\hat{P}$ is a coherent structure. Since $\hat{F}$ is not weakly amenable, $\hat{\tau} < \tau := \mu^{+Q'}$ (i.e. $\hat{F}$ only measures subsets of $\mu$ which are in $P^{\hat{\tau}}$). Let $\delta$ be maximal such that $\hat{\tau}$ is a cardinal in $P^{\hat{\delta}}$, and let $m := \text{degree}_{P^{\hat{\delta}}}(\mu)$. Assume $\text{ult}^*(P^{\hat{\delta}}, \hat{F})$ is wellfounded and let $\pi$ be the ultrapower map; note that $\pi$ is also the canonical lifting of $c_P$ to $P^{\hat{\delta}}$. Since $P^{\hat{\delta}}$ is sound above $\mu$ (it is a proper initial segment of the premouse $|P|$), then $\pi$ is $\Sigma_0^{m}$ preserving and $m$-cofinal (and $\Sigma_2^{k}$-preserving for $k < m$). Then $\text{ult}^*(P^{\hat{\delta}}, \hat{F})$ is a premouse, since $\pi^{\hat{\delta}} \upharpoonright \mu = \text{id}$ and the successor of $\mu$ is mapped continuously by $\pi$ (due to the $m$-cofinality of $\pi$).

**Definition 25.** Let $P$ be a protomouse with top extender $\hat{F}$, and $\delta$ be maximal such that $\hat{F}$ measures all subsets of $\text{cr}(\hat{F})$ which are in $P^{\hat{\delta}}$. If $\text{ult}^*(P^{\hat{\delta}}, \hat{F})$ is wellfounded, it is called the premouse associated with the protomouse $P$.

Finally, Lemma 16 implies that if $\text{cf}(\hat{\tau}) > \omega$ then $R$ exists (i.e. is wellfounded) and is fully $m$-iterable. The following summarizes all we will need to use about premice associated with protomice:
Lemma 26. \[ \bullet \; \text{ht}(P) = \lambda + \hat{R} \text{ where } \lambda \text{ is the largest cardinal in the coherent structure } P \]
\[ \bullet \; \text{Let } s \text{ be the sup of generators of } \hat{F}. \text{ Then } m = \text{degree}_{\hat{R}}(s) \text{ and } \hat{R} \text{ is sound above } s \]
\[ \bullet \; \text{The top (non weakly-amenable) extender of } P \text{ can be constructed from } R, \text{ since the range of the ultrapower map } \pi : \hat{Q}||\delta \to \hat{F} \hat{R} \text{ is } \hat{h}^{m+1}[\pi(\kappa) \cup \{r\}], \text{ where } r = \pi(p_{P||\delta}). \]
\[ \bullet \; \text{If } \text{cf}(\bar{\tau}) > \omega \text{ then } R \text{ is wellfounded and fully } m\text{-iterable.} \]

2.4. 0-pistol and the core model for non-overlapping extenders. The premice described in section 2.2 do not have overlapping extenders; i.e. the length of one extender is never $\geq$ the critical point of another extender. An object in the universe which resembles the premice described in the definition above, but has overlapping extenders, is called a $p$-mouse. It can be shown that $\omega$ is the first projection of any $p$-mouse, and that the transitive collapse of $h_M[\omega]$ does not depend on which $p$-mouse $M$ is used. This transitive collapse is called 0-pistol.

In the absence of 0-pistol, the theory of mice can be developed using linear iterations (i.e. without iteration trees). A weasel is a proper class mouse. A weasel $W$ is universal iff whenever $M$ is a set-size premouse which is coiterable with $W$, then the coiteration has length $< On$; if this holds then the $M$ side will always be simple.

In the absence of 0-pistol, the “core model for non-overlapping extenders” can be constructed. In this paper, $K$ refers to this core model. This core model is capable of having a single strong cardinal. Below we list a few key features of $K$ which will be used in this paper; see [10] for more details on 0-pistol and the construction and properties of $K$.

Definition 27. Let $W$ be a universal weasel and $F$ a $(\kappa, \nu)$ extender. Then $F$ is $W$-correct iff $F$ is weakly amenable with respect to $W$, ult$(W, F)$ is wellfounded, $\nu = o_{\text{ult}}(W, F)(\kappa)$ and $E^W \upharpoonright \nu = E^\text{ult}(W, F) \upharpoonright \nu$.

Lemma 28. (Essential facts about $K$): Assume 0-pistol does not exist, and let $K$ be the core model. Then:

1. $K$ is a universal weasel.
2. If $G$ is generic for some $\mathbb{P} \in V$ then 0-pistol does not exist in $V[G]$ and $K^{V[G]} = K$;
3. If $F$ is a $K$-correct extender then $F \in K$.
4. Assume $j : K \to K'$ is elementary with critical point $\kappa$, and let $F$ be the $(\kappa, j(\kappa))$-extender derived from $j$. Then $\nu := o^{K'}(\kappa) < j(\kappa)$, all generators of $F$ are $< \nu$, and $F|\nu = E^K_\nu$. 
Furthermore, $K'$ is a normal iterate of $K$ and $j$ is the corresponding iteration map.

**Corollary 29.** Assume $0$-pistol does not exist, $F$ is an extender with respect to $K$, and $\text{ult}(K,F)$ is wellfounded. Then $F \in K$.

A special case is when $\mathcal{W}$ is an ultrafilter on $P^K(\kappa)$ and $\mathcal{W}$ is normal with respect to $K$ (i.e. for every $f : \kappa \to \mathcal{W}$ in $K$, $\Delta f \in \mathcal{W}$). If $\text{ult}(K,\mathcal{W})$ is wellfounded and $j : K \rightarrow_{\mathcal{W}} K'$, then $\mathcal{W} \in K$ and generates the extender $E^{K}_{\sigma^{K'}(\kappa)}$.

**Lemma 30.** Assume $\mathcal{W}$ is an ultrafilter on $P^K(\kappa)$ which is $\kappa$-complete and weakly amenable with respect to $K$. Assume $\tau := \kappa^{+K}$ has uncountable cofinality. Then:

1. $\text{ult}(K|\tau,\mathcal{W})$ is wellfounded
2. If $\sigma^K(\mu) < \tau$ for every $\mu < \kappa$, then also $\text{ult}(K,\mathcal{W})$ is wellfounded.

**Proof.** Assume (1) fails, and let $\{f_n | n \in \omega\}$ witness the illfoundedness of $\text{ult}(K|\tau,\mathcal{W})$. Since $cf(\tau) > \omega$, there is some $\tilde{\tau} < \tau$ such that $\{f_n | n \in \omega\} \subseteq K|\tilde{\tau}$. Since $\mathcal{W}$ is weakly amenable with respect to $K$, then $\tilde{\mathcal{W}} := \mathcal{W} \cap K|\tilde{\tau}$ is an element of $K$, and in fact of the transitive $ZFC^-$ model $K|\tau$ (by acceptability). Then $K|\tau$ sees that $\text{ult}(K|\tilde{\tau},\tilde{\mathcal{W}})$ is illfounded; but then $\mathcal{W}$ is not complete with respect to $\omega$-sequences in $K$.

Item (2) follows from (1) and Corollary 19. □
3. Frequent extensions of embeddings lemma

This section is devoted to the proof of the Frequent Extensions of Embeddings Lemma for mice without overlapping extenders. The version presented here is more general than the versions in [10] and [5], though the main outline of the proof (originally due to Jensen) is the same. The generalizations include:

(G1): The mouse \( Q \) in the hypothesis of Lemma 31 does not necessarily have a largest cardinal;

(G2): We do not assume the mice \( M_X \) in the hypothesis of Lemma 31 are sound above \( \gamma_X \).

The part of the proof most affected by the generalizations (G1) and (G2) is the comparison of the premice which appear, e.g., in Claim 35. Lemma 20 and its corollary 23 are used in the comparison of those premice.

The generalization (G1) is important for the applications in this paper, e.g. the proof of Theorem 45.

Lemma 31. (Frequent Extensions of Embeddings Lemma) Assume:

- \( \kappa << \theta \) are regular cardinals and \( \kappa \geq \omega_2 \)
- \( \gamma \) is an ordinal \( \geq \kappa \)
- \( Q = J^{E_{\gamma}}_E \) is a mouse
- If there is some \( \mu < \gamma \) such that \( o^Q(\mu) = \gamma \) (i.e. \( Q \) has unboundedly many extenders with critical point \( \mu \)), we consider two cases:
  - \( \text{CASE (A)}: \gamma > \mu^+ \)
  - \( \text{CASE (B)}: \mu \) is the largest cardinal in \( Q \)

If (CASE (A)) holds, we require:

\[
\text{There is some } \lambda \in [\mu, \gamma) \text{ such that } cf(\lambda^+)^Q > \omega
\]

If (CASE (B)) holds, we make no additional requirements.

- \( S \) is a stationary set of weakly internally approachable \( X \in P_\kappa(H_\theta) \) such that \( (Q, \kappa, \gamma, ...) \in X \), and if (CASE (A)) holds we assume \( \lambda \in X \).

- For every \( X \in S \):
  - \( \pi_X : H_X \to H_\theta \) is the inverse of the Mostowski collapsing map of \( X \).
  - \( Q_X = \pi_X^{-1}(Q) \)
  - \( \sigma_X := \pi_X \upharpoonright Q_X \)
  - \( \gamma_X = \sigma_X^{-1}(\gamma) = ht(Q_X) \)
  - \( \tilde{\gamma}_X = sup(\sigma_X[\gamma_X]) \)
  - For every \( X \subseteq Y \) in \( S \), \( \sigma_{XY} = \sigma_Y^{-1} \circ \sigma_X \)
– $M_X$ is a premouse which end-extends $Q_X$ and $\gamma_X$ is regular in $\hat{M}_X$.
– There is an $n_X \in \omega$ such that $\gamma_X \leq \omega p^{n_X}_{\hat{M}_X}$, $R^{n_X}_{\hat{M}_X} \neq \emptyset$ if $n_X > 0$, $\gamma$ is not $\Sigma^1_{n_X-1}(\hat{M}_X)$-singularized, and $M_X$ is fully $n_X$-iterable.

Then for almost every $X \in S$:

- $\text{ult}^{n_X}(\overrightarrow{M_X}, \sigma_X)$ is wellfounded. Let $F_X$ denote the top predicate of $N_X$.
- If $N_X$ is a premouse–i.e. $F_X$ is total on $N_X$–then $N_X$ is fully $n_X$-iterable.
- If $N_X$ is not a premouse, then letting $\delta_X$ be maximal such that $F_X$ is a total extender on $N_X||\delta$, then $\text{ult}^*(\overrightarrow{N_X||\delta_X}, F_X)$ is fully $m_X$-iterable, where $m_X$ is the degree of $\mu_X$ in $\overrightarrow{N_X||\delta_X}$.

Note that if $\gamma = \kappa$, then we trivially have the conclusion, since in that case $\sigma_X \upharpoonright Q_X$ is the identity map for almost every $X \in S$ (i.e. $\hat{\gamma}_X = \gamma_X$). So from now on, assume $\kappa < \gamma$. We concentrate mostly on (CASE (A)), since this is the difficult case. Some parts of the proof will also vary depending on whether or not $\mu < \kappa$.

For almost every $X \in S$, we have $X \preceq (H_\theta, \in, \mu, \kappa, \ldots)$; without loss of generality assume this holds for every $X \in S$, and let $(\mu_X, \lambda_X, \ldots) = \pi^{-1}_X(\mu, \lambda, \ldots)$. By the $\kappa$-completeness of the nonstationary ideal on $P_\kappa(H_\theta)$, there is some $n \in \omega$ such that $n_X = n$ for stationarily many $X \in S$; without loss of generality suppose $n_X = n$ for every $X \in S$. Furthermore, since every $X \in S$ is weakly internally approachable, then $X \cap \text{Ord}$ is $\omega$-closed. To see this, just notice that whenever $a \subseteq X$ is a countable set of ordinals, weak internal approachability of $X$ implies there is some $b \in X$ such that $a \subseteq b \subseteq X$. Then $\sup(b) \in X$, so $a$ cannot be unbounded in $X \cap \text{Ord}$. Let $\beta$ be the least element of $X$ which is $\geq \sup(a)$. Then $\sup(\beta \cap b)$ is an element of $X$, and it is clear that $\sup(\beta \cap b) = \sup(a)$.

The $\omega$-closure of $X \cap \text{Ord}$ implies that:

(2) \quad $\pi_X$ is continuous on $\text{cof}(\omega)$

If (CASE (A)) of Lemma 31 holds, then (2) guarantees:

(3) \quad $\text{cf}(\lambda^+_{X^Q_X}) > \omega$
3.1. Wellfoundedness of $\text{ult}^n(M_X, \sigma_X)$. We first prove that for almost every $X \in S$, $\text{ult}^n(M_X, \sigma_X)$ is wellfounded; note that this ultrapower uses good $\Sigma_1^{(n-1)}(M_X)$ functions. Suppose this fails; WLOG assume it fails for every $X \in S$. For each $X \in S$, let $\langle (f_i^X, r_i^X, \alpha_i^X) \mid i \in \omega \rangle$ witness the illfoundedness of the canonical lifting, where:

- $f_i^X$ is a good $\Sigma_1^{(n-1)}(M_X)$ function in parameter $r_i^X$, and $\text{dom}(f_i^X) < \gamma_X$. If [CASE (B)] holds—i.e. $\mu$ is the largest cardinal in $Q$—then we can without loss of generality assume that $\text{dom}(f_i^X) = \mu_X$ for every $X \in S$ and $i \in \omega$

- $\alpha_i^X < \sigma(\text{dom}(f_i^X))$

- For every $i \in \omega$, $\langle \alpha_i^{X_1}, \alpha_i^X \rangle \in \sigma(\text{dom}(f_i^X))$, where $u_i^X := u_i^M_X : f_i^X \in f_i^X$

(see Definition 3).

Claim 32. Without loss of generality, we may assume that the diagram is minimal; i.e. that for every $X \in S$ there is no $\zeta_X < \text{ht}(Q_X)$, $m_X \in \omega$, and premouse $R_X$ such that:

- $\zeta_X$ is regular in $\hat{R}_X$, $\zeta_X \leq \omega \rho_{R_X}^{m_X}$ and $\zeta_X$ is not $\Sigma_1^{(m_X - 1)}(\hat{R}_X)$-singularized, and $R_{R_X}^{m_X} \neq \emptyset$ if $m_X > 0$.

- $R_X$ is $m_X$-iterable.

- $\text{ult}^{m_X}(R_X, \sigma_X \upharpoonright \zeta_X)$ is illfounded

Proof. If this does not hold for every $X \in S$, let $T \subseteq S$ be the collection of $X$ where it fails. If $T$ is nonstationary, just replace $S$ with $S - T$. If $T$ is stationary, for every $X \in T$ select $\zeta_X$ to be the minimal object witnessing the first bullet above. Using the Fodor Lemma with the regressive map $X \mapsto \sigma_X(\zeta_X)$ on $T$, there is some stationary $T' \subseteq T$ such that for every pair $X, Y \in T'$ where $X \subseteq Y$, then $\sigma_X(Y)(\zeta_X) = \zeta_Y$. Then replace $S$ with $T'$ and then for every $X \in T'$ replace the $\sigma_X$ with $\sigma_X \upharpoonright \zeta_X$. Then the resulting system is minimal. \qed

For each $X \in S$ let $A_X \prec (H_\theta, \in)$ be countable such that $M_X \subseteq A_X$ and $\{r_i^X \mid i \in \omega\} \subseteq A_X$. If $n > 0$, recall we assume that $R_{M_X}^n \neq \emptyset$; in this case fix some $q_X \in R_{M_X}^n$. Let $\sigma_X^* : M_X^* \to A_X \prec H_\theta$ be the inverse of the collapsing map, and $(M_X^*, \gamma_X^*, Q_X^*, f_i^{X^*}, r_i^{X^*}, q_X^*, ...) = (\sigma_X^*)^{-1}(M_X, \gamma_X, Q_X, f_i^X, r_i^X, q_X, ...)$. By elementarity of $\sigma_X^*$ and our assumptions on $M_X$:

- $M_X^*$ is $n$-iterable

- $\gamma_X^* \leq \omega \rho_{M_X^*}^n$ and $\gamma_X^*$ is not $\Sigma_1^{(n-1)}(M_X^*)$-singularized

- $R_{M_X}^n \neq \emptyset$ if $n > 0$
Claim 33. For almost every \( X \in S \), \( \sigma_X^* \) maps \( \gamma_X^* \) cofinally to its image, and so \( \gamma_X \) has countable cofinality.

Proof. Suppose this fails at stationarily many \( X \in S \). Let \( \tilde{\gamma}_X < \gamma_X \) be the cofinal image of \( \gamma_X^* \) under \( \sigma_X^* \). By the interpolation lemma, \( \tilde{P}_X := \text{ult}^n(\tilde{M}_X, \sigma_X^* \upharpoonright \gamma_X^*) \) is wellfounded; let \( \tilde{\sigma}_X^* : \tilde{M}_X \to_{\Sigma_0^{(n)}} \tilde{P}_X \) be the \( n \)-lifting and \( \sigma_X' : \tilde{P}_X \to \tilde{M}_X \) be the map from the Interpolation Lemma. By (1) and the facts from the preliminaries section, \( \tilde{\sigma}_X^* \) is \( n \)-cofinal, \( \tilde{\sigma}_X^* \) maps \( \gamma_X^* \) continuously (i.e. to \( \gamma_X^* \)), and \( \sigma_X' \) is \( \Sigma_0^{(n)} \) preserving. Also, if \( n > 0 \) then \( \tilde{\sigma}_X^*(q_X^*) \in R_{\tilde{P}_X}^n \).

For \( i \in \omega \), let \( \tilde{f}_i^X := \tilde{\sigma}_X^*(f_i^X) \). Since \( \sigma_X^* \upharpoonright \gamma_X^* = \text{id} \) and is \( \Sigma_0^{(n)} \) preserving, for every \( i \in \omega \) we have \( u_{\tilde{P}_X}^{\tilde{f}_i^X} = u_{\tilde{M}_X}^{\tilde{M}_X} \). In particular, \( \text{ult}^n(\tilde{P}_X, \sigma_X \upharpoonright \gamma_X^*) \) is not wellfounded. We will now obtain a contradiction by showing that either \( \tilde{P}_X \), or a suitable replacement for it, satisfies the properties of the \( R_X \) from Claim 32. This will contradict Claim 32 since \( \tilde{\gamma}_X < \gamma_X \).

If \( P_X \) is a premouse—i.e. its top predicate is weakly amenable—then Lemma 15 implies that \( P_X \) is \( n \)-iterable. Thus the system \( \langle (\tilde{P}_X, \sigma_X \upharpoonright \gamma_X^*) \rangle_{X \in S} \) constitutes a counterexample to the Frequent Extensions of Embeddings Lemma, contradicting Lemma 32.

Now assume that \( \tilde{P}_X \) is not a premouse—i.e. it has a top predicate that is not weakly amenable. We will show that the premouse \( R_X \) associated with \( P_X \) (see Definition 25) is wellfounded for each \( X \), and that this system of \( R_X \) contradicts Lemma 32 (since \( \tilde{\gamma}_X < \gamma_X \)). Since this situation can only occur if \( n = 0 \), then WLOG we can assume:

\[ \tilde{f}_i^X \text{ is an element of } \tilde{P}_X \text{ for every } i \in \omega. \]

Let \( \pi : \tilde{M}_X \to_{E_{\tilde{M}_X}^{\tilde{P}_X}} \tilde{M}_X', \delta_X \) be maximal such that \( \tilde{F}_X := E_{\tilde{M}_X}^{\tilde{P}_X} \) is total in \( \tilde{P}_X \upharpoonright \delta_X = Q_X \upharpoonright \delta_X \). Let \( R_X' := \pi(Q_X \upharpoonright \delta_X) \); note \( R_X' \) is a proper initial segment of \( M_X' \) and so is fully iterable. Let \( m = \text{degree}_{Q_X \upharpoonright \delta_X}(\mu_X) \). Define the map \( j : \text{ult}^n(Q_X \upharpoonright \delta_X, \tilde{F}_X) \to \tilde{R}_X' \) by \( [\eta, f]_{\tilde{F}_X}^{Q_X \upharpoonright \delta_X} \mapsto \pi(f)(\sigma_X' (\eta)) \), where \( f \) is a good \( \Sigma_1^{(m-1)}(Q_X \upharpoonright \delta_X) \) function with domain \( \mu_X \) and \( \eta < lh(\tilde{F}) \). It is straightforward to check that \( j \) is well-defined and \( \Sigma_0 \) preserving, so the premouse \( R_X \) associated with \( \tilde{P}_X \) is wellfounded. In fact, \( j \) is \( \Sigma_0^{(m)} \) preserving, which implies by Lemma 14 that \( R_X \) is fully \( m \)-iterable. This yields a contradiction to Claim 32, since if \( 5 \) implies that the functions \( f_i^X \) are elements of \( \tilde{R}_X \) and so \( \text{ult}(\tilde{R}_X, \sigma_X \upharpoonright \tilde{\gamma}_X^*) \) is illfounded. \( \square \)
Since each $X \in S$ is weakly internally approachable, there is some $Z_X$ which is an element and subset of $X$ such that $\sigma_X \circ \sigma_X^* [Q_X^*] \subseteq Z_X$. By the Fodor Lemma for $P_\kappa(H_\theta)$ we may assume WLOG that $Z_X$ is the same for every $X \in S$. Fix some $X_0 \in S$; then for every $X \in S$ such that $X_0 \subseteq X$, $\text{rng}(\sigma_X^*) \subseteq \text{rng}(\sigma_{X_0, X})$. From now on, $Q_{X_0}$ will be denoted $Q_0$, and WLOG we assume that every $X \in S$ is a superset of $X_0$. For each $X \in S$, $\sigma_{0, X} \circ \sigma_X^*$ is a cofinal map from $Q_X^*$ to $Q_0$. Using an interpolation-like argument and (1):

**Lemma 34.**  

- $\text{ult}^n(M_X^*, \sigma_{0, X} \circ \sigma_X^* \mid \gamma_X^*)$ is wellfounded; let $\tilde{\sigma}_X^*$:
  
  $\tilde{M}_X^* \rightarrow \tilde{P}_X^*$ be the corresponding ultrapower map.

- $\gamma_0 = \tilde{\sigma}_X^*(\gamma_X^*) < \omega_0^\tilde{P}_X^*$ and $R^n_{\tilde{P}_X^*} \neq \emptyset$.

- There is a $\Sigma_0^{(n)}$ map $\tilde{\sigma}_X$ from $\tilde{P}_X$ to $\tilde{M}_X$ which extends $\sigma_{0, X}$.

For each $i \in \omega$ and $X \in S$, let $f^X_i := \tilde{\sigma}_X^*(f_X^i)$ and $u^X_{\tilde{P}_X} := u^X_{\tilde{P}_X, \tilde{f}_i, i, j}$. Since $\tilde{\sigma}_X$ extends $\sigma_{0, X}$, then for every $i \in \omega$:

\[
\tilde{\sigma}_X^*(u_i^{\tilde{P}_X}) = \sigma_{0, X}(\tilde{\sigma}_X^*(u_i^{M_X^*})) = \sigma_X^*(u_i^{M_X^*}) = u_i^{M_X}.
\]

For every $X \in S$, let $\tilde{q}_X := \tilde{\sigma}_X^*(q_X^*)$. $\tilde{q}_X$ is an element of $R^n_{\tilde{P}_X}$ and $\tilde{\sigma}_X(\tilde{q}_X) = q_X$.

We first take care of the case where $n > 0$.

3.1.1. When $n > 0$. For this section, assume $n > 0$. Since $\tilde{\sigma}_X$ is $\Sigma_0^{(n)}$ preserving where $n > 0$, then the $\Pi_2$ property of being a premouse is preserved downward by $\tilde{\sigma}_X^*$. So $\tilde{P}_X$ is a premouse; let $\tilde{M}_X$ denote the bookkeeping premouse of which $\tilde{P}_X$ is the expansion. Lemma 14 guarantees that $\tilde{M}_X$ is $n$-iterable.

**Claim 35.** There is a pair $X \subset Y$ in $S$ such that:

1. In the n-coiteration of $\tilde{M}_X$ with $\tilde{M}_Y$, the $\tilde{M}_X$ side is simple
2. $o^{\tilde{M}_X}(\mu_0) \leq o^{\tilde{M}_Y}(\mu_0)^7$
3. $\{\alpha^X_i \mid i \in \omega\} \subseteq \text{rng}(\sigma_Y)$

**Proof.** Construct a sequence $\langle Z_k \mid k \in \omega \rangle$ as follows: pick $Z_0$ such that $o^{\tilde{M}_0}(\mu_0)$ is minimal. Given $Z_k$, choose $Z_{k+1}$ such that $o^{\tilde{M}_{Z_k+1}}(\mu_0)$ is minimal among all $Z \in S$ which are “sufficiently large” with respect to each of $Z_0, \ldots, Z_k$; i.e. minimal among the set $\{o^{\tilde{M}_k}(\mu_0) \mid Z \in S\}$ and for each $j \leq k$, $\{\alpha^Z_i \mid i \in \omega\} \subset \text{range}(\sigma_Z)$. This construction ensures that $(o^{\tilde{M}_k}(\mu_0) \mid k \in \omega)$ is monotone increasing. Since the $<^n$ is

\[\text{Alternatively, the canonical lifting of } \sigma_{0, X}^{-1} \circ \sigma_X^* \text{ is } \Sigma_2^{(\sigma^{-1})} \text{ preserving, so premousehood of } \tilde{M}_X^* \text{ is preserved upward to } \tilde{P}_X \text{ by this map.}\]

\[\text{This requirement is irrelevant if } \text{CASE (B) holds.}\]
wellfounded, there is some $k^* \in \omega$ such that $M_{Z_{k^*+1}} \geq^n M_{Z_{k^*}}$. Then the pair $M_{Z_{k^*}}, M_{Z_{k^*+1}}$ satisfies the requirements of the claim. □

Let $X \subset Y$ be as in Claim 35. Let $S$ denote the last premouse in the $n$-coiteration of $M_X$ with $M_Y$, $\bar{T}_X$ and $\bar{T}_Y$ the $n$-iterations on the $X, Y$ sides respectively, and $\bar{\pi}_X$ the $n$-iteration map on the (simple) $M_X$ side. By Lemma 11 the map $\bar{\sigma}_Y : \bar{M}_Y \to \Sigma_0^{(n)} \bar{M}_Y$ can be used to copy $\bar{T}_Y$ to a normal $n$-iteration $\bar{T}_Y$ of $M_Y$, and all the copy maps are still $\Sigma_0^{(n)}$-preserving. Let $S^*$ be the last iterate of $\bar{T}_Y$, and $j^* : \hat{S} \to \Sigma_0^{(n)} \hat{S}^*$ be the last copy of $\bar{\sigma}_Y$. The following diagram describes the situation; wavy lines indicate possibly non-simple $n$-iterations:

$$
\begin{array}{ccc}
M_Y & \xrightarrow{\bar{\sigma}_Y} & M_Y \\
\bar{T}_Y & \nearrow \bar{T}_Y & \searrow \bar{T}_Y \\
M_X & \xrightarrow{\bar{\pi}_X} & \hat{S} & \xrightarrow{j^*} S^*
\end{array}
$$

Then for every $i \in \omega$, (6) implies:

$$
\langle \alpha^X_{i+1}, \alpha^X_i \rangle \in \sigma^X(u^M_X) = \sigma_X(\sigma_{0,X}(u^M_X))
$$

So applying $\sigma^{-1}_Y$ yields

$$
\langle \sigma^{-1}_Y(\alpha^X_{i+1}), \sigma^{-1}_Y(\alpha^X_i) \rangle \in \sigma^{-1}_Y(\sigma_X(\sigma_{0,X}(u^M_X))) = \sigma_{0,Y}(u^M_X) = j^*(u^M_X)
$$

The last equality in (8) is due to the fact that the first iteration index of $\bar{T}_Y$ is $\geq ht(Q_0)$ (because $Q_0$ is a common initial segment of $M_X, M_Y$), so $j^* \upharpoonright Q_0 = \bar{\sigma}_Y \upharpoonright Q_0 = \sigma_{0,Y}$.

The remainder of the proof varies slightly, depending on whether CASE (A) or CASE (B) in Lemma 31 holds.

3.1.2. When CASE (A) holds (i.e. $\mu^+Q < \gamma$). Recall in this case, the hypothesis of Lemma 31 states that there is some $\lambda \in [\mu, \gamma]$ such that $\lambda^+Q$ has uncountable cofinality. So (3) holds at $Q_0$; i.e. $\lambda^+Q_0$ has uncountable cofinality (in $V$). Since $\lambda_X \in [\mu_0, \gamma_0]$ and $M_X, M_Y$ agree below $\gamma_0$, then Corollary 23 implies that $M_X$ agrees with $M_Y$ below $\min(o^M_X(\mu_0), o^M_Y(\mu_0)) = o^M_X(\mu_0)$; the last equality is due to the fact that $X, Y$ are as in Claim 35. This means that in the $n$-coiteration of $M_X$ with $M_Y$:

$$
\text{The } M_X \text{ side is simple and (strictly) above } \gamma_0
$$

(We cannot rule out the possibility that $o^M_X(\mu_0), o^M_Y(\mu_0)$ are distinct; i.e. we cannot always guarantee that both sides of $M_X$ vs. $M_Y$ are above $\gamma_0$.)
So \( \bar{\pi}_X \upharpoonright Q_0 = id \), and for each \( i \in \omega \) we have \( u_i^{\bar{M}_X} = \bar{\pi}_X(u_i^{\bar{M}_X}) \). This combined with (8) implies that for every \( i \in \omega \):

\[
(10) \quad \langle \sigma_{\bar{Y}}^{-1}(\alpha_i^{X+1}), \sigma_{\bar{Y}}^{-1}(\alpha_i^X) \rangle \in j^*(\bar{\pi}_X(u_i^{\bar{M}_X}))
\]

Since \( j^* \circ \bar{\pi}_X \) is \( \Sigma_0^{(n)} \) preserving, the functionally absolute definitions of the good \( \Sigma_1^{(n-1)}(\bar{M}_X) \) functions \( \bar{f}_i^X \) can be copied via \( j^* \circ \bar{\pi}_X \), and so for every \( i \in \omega \):

\[
(11) \quad \bar{j}^*(\bar{\pi}_X(u_i^{\bar{M}_X})) = \{ (\xi, \eta) \in j^*(\bar{\pi}_X(\text{dom}(\bar{f}_i^X) \cup \text{dom}((\bar{f}_i^X))) | j^*(\bar{\pi}_X(\bar{f}_i^X)))((\xi) \in j^*(\bar{\pi}_X(\bar{f}_i^X)))((\eta)) \}
\]

Then (10) and (11) imply \( \langle j^*(\bar{\pi}_X(\bar{f}_i^X))(\bar{\sigma}_{\bar{Y}}^{-1}(\alpha_i^X)) | i \in \omega \rangle \) is an infinite decreasing chain of ordinals in \( \bar{S}^* \), a contradiction.

3.1.3. When \[ \text{CASE (B)} \] holds (i.e. \( \mu_{i+Q_0} = \gamma_0 \)). Recall in this case we assumed without loss of generality that the domain of each \( f_i^X \) was \( \mu_X \). Now (8) may not hold now; but \( \bar{M}_X, \bar{M}_Y \) coiterate above \( \mu_0 \), so \( u_i^{\bar{M}_X} = \bar{\pi}_X(u_i^{\bar{M}_X}) \cap \mu_0 \). This, combined with (8) implies:

\[
(12) \quad \langle \sigma_{\bar{Y}}^{-1}(\alpha_i^{X+1}), \sigma_{\bar{Y}}^{-1}(\alpha_i^X) \rangle \in j^*(\bar{\pi}_X(u_i^{\bar{M}_X}) \cap \mu_0)
\]

So again we have that \( \langle j^*(\bar{\pi}_X(\bar{f}_i^X))(\bar{\sigma}_{\bar{Y}}^{-1}(\alpha_i^X)) | i \in \omega \rangle \) is an infinite decreasing chain of ordinals in \( \bar{S}^* \), a contradiction.

3.1.4. When \( n = 0 \). For this subsection, we assume \( n = 0 \), so WLOG the \( \bar{f}_i^X \) are elements of \( \bar{P}_X \). If \( \bar{P}_X \) is a premouse, then the argument is exactly the same as when \( n > 0 \); there is a pair \( X, Y \) as in Claim [35] where in this case we perform 0-coiterations. Let \( \bar{I}_X, \bar{I}_Y, \bar{\pi}_X, \bar{j}^*, \bar{S}, \bar{S}^* \) be as in section [3.1.1]. In the present case, \( j^* \circ \bar{\pi}_X \) is \( \Sigma_0 \) preserving, which is sufficient to obtain a descending chain of ordinals in \( \bar{S}^* \) since the functions \( \bar{f}_i^X \) are elements of \( \bar{M}_X \).

If \( \bar{P}_X \) is a protomouse, let \( \delta_X \) be maximal such that \( E_{\bar{P}_X}^{\bar{P}_X} \) is total for \( \bar{P}_X \upharpoonright \delta_X = Q_0 \upharpoonright \delta_X \), and let \( \bar{\tau}_X \) be the successor of \( \mu_0 \) in \( Q_0 \upharpoonright \delta_X \). Note that since \( |Q_0| < \kappa \), there is a pair \( (\delta, \bar{\tau}) \) such that for stationarily many \( X, (\bar{\tau}_X, \delta_X) = (\bar{\tau}, \delta) \). WLOG assume this holds for every \( X \in S \).

Let \( \pi : \bar{M}_X \to E_{\bar{P}_X}^{\bar{M}_X} \bar{M}_X \) and \( \bar{Q}_X := \bar{\sigma}_X(Q_0 \upharpoonright \delta) \). Let \( \bar{N}_X = \pi(\bar{Q}_X) \); this is fully iterable. Then \( \bar{R}_X := \text{ult}(\bar{Q}_X \upharpoonright \delta_X, E_{\bar{P}_X}^{\bar{P}_X}) \) (the premouse associated with \( \bar{P}_X \)) can be \( \Sigma_0 \)-embedded into the mouse \( \bar{N}_X \) via a copy of \( (\bar{\sigma}_X \upharpoonright Q_0 \upharpoonright \delta_X, \bar{\sigma}_X \upharpoonright lh(E_{\bar{P}_X}^{\bar{P}_X})) \); denote this copy map by \( j_X \). So \( \bar{R}_X \) is wellfounded and 0-iterable, and \( j_X \) agrees with \( \bar{\sigma}_X \) below.
$lh(E_\text{top}(P_X))$ (the largest cardinal in $\bar{P}_X$). In particular:

\begin{equation}
\text{(13)} \quad \text{For every } X, j_X \upharpoonright Q_0 = \sigma_{0,X}
\end{equation}

Note that since $n = 0$, each $\bar{f}_i^X$ is an element of $\bar{P}_X$ and thus of $\bar{R}_X$.

So \text{(13)} implies:

\begin{equation}
\text{(14)} \quad \text{For every } X \text{ and every } l \in \omega, j_X(u_i^{\bar{R}_X}) = \sigma_{0,X}(u_i^{\bar{P}_X}) = u_i^{M_X}
\end{equation}

As in Claim (35), there is a pair $X, Y$ in $S$ such that:

\begin{enumerate}
\item In the 0-coiteration of $\bar{R}_X$ with $\bar{R}_Y$, the $\bar{R}_X$ side is simple
\item $o^{\bar{R}_X}(\mu_0) \leq o^{\bar{R}_Y}(\mu_0)$
\item $\{\alpha_i^X | i \in \omega\} \subseteq \text{rng}(\sigma_Y)$
\end{enumerate}

The rest of the proof is almost identical to the last section. Copy the $R_Y$ side of the 0-coiteration to a 0-iteration of $N_Y$ (via the map $j_Y$), let $j^* : \hat{S} \rightarrow \hat{S}^*$ be the last copy map, and $\pi_X$ the iteration map on the (simple) $\bar{R}_X$ side of the 0-coiteration. The following diagram depicts the situation:

\begin{equation}
\begin{array}{c}
\bar{R}_Y \xrightarrow{j_Y} N_Y \\
\downarrow \pi_Y \quad \downarrow \pi_Y \\
\bar{R}_X \xrightarrow{\pi_X} \hat{S} \xrightarrow{j^*} \hat{S}^*
\end{array}
\end{equation}

Then exactly as in the last section, $\langle j^*(\pi_X(\bar{f}_i^X))(\sigma_Y^{-1}(\alpha_i^X)) | i \in \omega \rangle$ is a decreasing chain of ordinals in $\hat{S}^*$, a contradiction.

**3.2. Iterability of the ultrapowers.** We showed in the last section that for almost every $X \in S$, $\text{ult}^{n_X}(\bar{M}_X, \sigma_X)$ is wellfounded. Without loss of generality, assume this holds for every $X \in S$. Define:

\begin{equation}
L_X := \text{ult}^{n_X}(\bar{M}_X, \sigma_X)
\end{equation}

and for simplicity let $\sigma_X$ also denote the extension of the original map; so $\sigma_X : \bar{M}_X \rightarrow \Sigma^{(n_X)} L_X$. Due to the fine-structural assumptions on $\bar{M}_X$, this lifting is $n_X$-cofinal, $\bar{\gamma}_X = \sigma_X(\gamma_X)$, $\bar{\gamma}_X \leq \omega \rho^X_{L_X}$, and if $n_X > 0$ then $\sigma_X(q_X) \in R^{n_X}_{L_X}$ (recall $q_X$ is our chosen element of $R^{n_X}_{M_X}$).

We need to show that for almost every $X \in S$:

- If $L_X$ is a premouse then it is $n_X$-iterable
- If $L_X$ is a protomouse whose top predicate is a non-weakly-amenable extender with critical point $\mu$ then the usual premouse $\bar{R}_X$ associated with it is wellfounded and normally $m_X$-iterable, where $m_X = \text{degree}_{\bar{R}_X}(\mu)$. 
Suppose this fails for stationarily many \( X \in S \); without loss of generality we assume it fails for every \( X \in S \). Using the \( \kappa \)-completeness of the nonstationary ideal on \( P_\kappa(H_\theta) \), without loss of generality we assume that \( n_X \) is fixed at some \( n \) for every \( X \in S \).

We first address the case when \( L_X \) is a premouse for stationarily many \( X \in S \).

3.2.1. When \( L_X \) is a premouse. For this section, assume \( L_X \) is a premouse for every \( X \in S \). Similarly to Lemma 3.2, assume WLOG that the current diagram is minimal:

Claim 36. WLOG we can assume for every \( X \in S \): whenever \( \zeta_X < \gamma_X \) is a regular cardinal in some premouse \( R_X \) such that \( \langle R_X, \sigma_X \mid \zeta_X, ... \mid X \in S \rangle \) satisfies the hypothesis of the Frequent Extensions of Embeddings Lemma, then the conclusions of that lemma hold for this system.

For each \( X \in S \):

1. Let \( N_X \) denote the bookkeeping premouse associated with \( L_X \); i.e. \( L_X = \widehat{N}_X \).
2. Fix a countable elementary substructure \( A_X \) of \( H_\theta \) such that \( (L_X, \sigma_X(q_X), ...) \in A_X \), and let \( \pi_X^* : H_X^* \to H_\theta \) be the inverse of the collapsing map for \( A_X \). Let \( (L_X, N_X, ...) = (\pi_X^*)^{-1}(L_X, N_X, ...) \).

So by elementarity of \( \pi_X^* \), \( N_X \) is a (bookkeeping) premouse which is not \( n \)-iterable. Furthermore, if \( n > 0 \) then \( (\pi_X^*)^{-1}(\sigma_X(q_X)) \in R^n_{\widehat{N}_X} \).
3. Let \( \{z_X^i \mid i \in \omega \} \) enumerate \( \widehat{N}_X^* \).
4. Select \( \{(f_X^i, \alpha_X^i, r_X^i) \mid i \in \omega \} \) where \( \pi_X^*(z_X^i) = \sigma_X(f_X^i)(\alpha_X^i) \) for every \( i \in \omega \), and \( f_X^i \) is a good \( \Sigma_1^{(n-1)}(\hat{M}_X) \) function in the parameter \( r_X^i \). By the \( n \)-cofinality of the map \( \sigma_X \), if \( z_X^i \) is in the \( n \)-th reduct of \( \widehat{N}_X^* \) then we can take \( f_X^i \) to be an element of the \( n \)-th reduct of \( \hat{M}_X \).
5. Construct countable \( M_X^* \) and fully elementary maps \( \sigma_X^* : \widehat{M}_X^* \to \widehat{M}_X \) exactly as section 3.1, except here make sure that \( \{r_X^i \mid i \in \omega \} \subseteq \text{rng}(\sigma_X^*) \).
6. Letting \( \tilde{\gamma}_X \) be \( \text{sup}(\text{range}(\sigma_X^*) \cap \gamma_X) \), note that the supremum of \( \sigma_X[\tilde{\gamma}_X] \) is \( \geq \text{sup}_{i < \omega} \alpha_i^X \).

Claim 37. For almost every \( X \in S \), \( \sigma_X^* \) maps \( \gamma_X^* \) cofinally to \( \gamma_X \).

Proof. Assume this fails for stationarily many \( X \in S \), and let \( \tilde{\gamma}_X^* \) be the cofinal image of \( \gamma_X^* \) under \( \sigma_X^* \). Construct \( \tilde{P}_X \) and the corresponding
map \( \sigma'_X : \tilde{P}_X \rightarrow \Sigma_0^{(n)} \tilde{M}_X \) exactly as in claim \[33\]. Since \( \{ r^X_i \mid i \in \omega \} \subseteq \text{rng}(\sigma'_X) \) then \( \tilde{P}_X \) has “copies” of the \( f^X_i \) which we’ll denote by \( \tilde{f}^X_i \) (for \( i \in \omega \)); more precisely, \( \tilde{f}^X_i \) is the good \( \Sigma_1^{(n-1)}(\tilde{P}_X) \) function in the parameter \( (\sigma'_X)^{-1}(r^X_i) \) which has the same definition.

First suppose that \( \tilde{P}_X \) is a premouse for almost every \( X \in T \). Then by the minimality assumption of the diagram, the system \( \langle \tilde{P}_X, \sigma_X \upharpoonright \tilde{\gamma}_X^* \mid X \in T \rangle \) satisfies the conclusions of the Frequent Extensions of Embeddings Lemma. Let \( \bar{\sigma}_X : \tilde{P}_X \rightarrow \Sigma_0^{(n)} \tilde{L}_X^\prime \) be the lifting guaranteed by the lemma. Then \( \tilde{L}_X^\prime \) is a premouse since we are assuming \( n > 0 \), and \( \tilde{L}_X^\prime \) is \( n \)-iterable.

Define \( j : \tilde{N}_X^* \rightarrow \tilde{L}_X^\prime \) by \( j(z^X) := \bar{\sigma}_X(\tilde{f}^X_i)(\alpha^X_i) \); this is well-defined by the comments above, i.e., \( \alpha^X_i \) really is in the domain of \( \bar{\sigma}_X(\tilde{f}^X_i) \). The map \( j \) is \( \Sigma_0^{(n)} \) preserving, so Lemma \[14\] guarantees that \( \tilde{N}_X^* \) is \( n \)-iterable, contrary to our assumptions. That \( j \) is \( \Sigma_0^{(n)} \) preserving follows from a straightforward computation, using \( \Sigma_0^{(n)} \) Los’ Theorem, the fact that \( \sigma'_X \) is \( \Sigma_0^{(n)} \) preserving, and the fact that if \( z^X \) is in the \( n \)-th reduct of \( \tilde{N}_X^* \) then WLOG we can take \( f^X_i \) to be an element of the \( n \)-th reduct of \( \tilde{M}_X \) (and thus \( \tilde{f}^X_i \) is an element of the \( n \)-th reduct of \( \tilde{P}_X \)). A similar computation is given in Claim \[38\] below.

Now suppose \( \tilde{P}_X \) is a protomouse (for stationarily many \( X \in T \)); this can only happen if \( n = 0 \). As in the proof of Claim \[33\] the premouse \( R_X \) associated with \( \tilde{P}_X \) is wellfounded and satisfies the hypothesis of Claim \[36\]. Let \( \tilde{L}_X^\prime = \text{ult}^*(\bar{R}_X, \sigma_X \upharpoonright \tilde{\gamma}_X^*) \).

\( \tilde{L}_X^\prime \) is a premouse: otherwise \( \mu^{\tilde{R}_X}_{X^\prime} < \tilde{\gamma}_X^* \) and \( \sigma_X \upharpoonright \tilde{\gamma}_X^* \) maps this ordinal non-continuously. But then \( \mu^{\tilde{R}_X}_{X^\prime} = \mu^{\tilde{M}_X}_{X^\prime} \) (since \( cr(\sigma'_X) = \tilde{\gamma}_X^* \)) and so \( \bar{\sigma}_X \) (which extends \( \sigma_X \)) would yield that \( \tilde{L}_X \) is a protomouse, contrary to the assumption in this section.

Let \( \bar{\sigma}_X : \tilde{R}_X \rightarrow \tilde{L}_X^\prime \) be the canonical lifting. Finally, construct a map \( j : \tilde{N}_X^* \rightarrow \tilde{L}_X^\prime \) as in the earlier part of this proof (note that the earlier definition of \( j \) still makes sense here, because \( \tilde{f}^X_i \) is an element of \( \tilde{R}_X \) and thus of \( R_X \)). It is straightforward to check that this map is \( \Sigma_0 \) preserving, which contradicts the fact that \( \tilde{N}_X^* \) is not \( 0 \)-iterable. \[ \square \]

Using the same arguments as in section \[3.1\] there is a \( Q_0 \) such that for stationarily many \( X \), \( \sigma_0^{-1}_X \circ \sigma^X_X \) is a total \( \Sigma_0 \) cofinal map from \( Q^*_X \to Q_0 \).

For such \( X \), as in Lemma \[34\] construct \( \bar{\sigma}_X : \tilde{P}_X \rightarrow \Sigma_0^{(n)} \tilde{M}_X \), where \( \tilde{P}_X = \text{ult}^*(M^*_X, \sigma_0^{-1}_X \circ \sigma^X_X) \) and \( \bar{\sigma}_X \) extends \( \sigma_{0,X} \).
As in section 3.1, we break the proof into cases according to the value of \( n \).

### 3.2.2. When \( n > 0 \)

Since we’re assuming \( n > 0 \), \( \bar{\sigma}_X \) is \( \Sigma_2 \) preserving and \( \bar{P}_X \) is a premouse. Let \( \bar{M}_X \) be its associated bookkeeping premouse; i.e. \( P_X = \bar{M}_X \). Lemma 14 guarantees that \( \bar{M}_X \) is \( n \)-iterable.

Let \( \bar{f}^X_i \) denote the “copy” of \( f^X_i \) over \( \bar{M}_X \); more precisely, \( \bar{f}^X_i \) is the good \( \Sigma_1^{(n-1)}(\bar{M}_X) \) function in the parameter \( (\bar{\sigma}_X)^{-1}(r_X) \) which has the same definition as \( f^X_i \) has over \( \bar{M}_X \) (in the parameter \( r_X \)). As in Claim (35) there is a pair pair \( X \subset Y \) in \( S \) such that:

- In the \( n \)-coiteration of \( \bar{M}_X \) with \( \bar{M}_Y \), the \( \bar{M}_X \) side is simple
- \( o^{\bar{M}_X}(\mu_0) \leq o^{\bar{M}_Y}(\mu_0) \)
- \( \{\alpha_i^X | i \in \omega\} \subseteq \text{rng}(\sigma_Y) \)

Let \( \bar{I}_X, \bar{I}_Y \) denote the the \( \bar{M}_X, \bar{M}_Y \) sides of their \( n \)-coiteration (respectively), let \( \bar{\pi}_X : \bar{M}_X \to \bar{S} \) denote the iteration map on the (simple) \( \bar{M}_X \) side, \( \bar{I}_Y \) be the copy of \( \bar{I}_Y \) to \( M_Y \) via \( \bar{\sigma}_Y \) which exists by Lemma 11, and \( j^* : \bar{S} \to \bar{S}^* \) be the last copy map. Define \( k : \bar{N}^*_X \to \bar{S}^* \) by \( z^X_i \mapsto j^* \circ \bar{\pi}_X(\bar{f}^X_i)(\sigma_Y^{-1}(\alpha_i^X)) \). In section 3.1 we found a decreasing chain of ordinals in \( S^* \); but here we will use \( S^* \) and \( k \) to show that \( N^*_X \) is \( n \)-iterable to yield a contradiction and complete the proof for section 3.2.2. The situation is depicted in the following diagram, where wavy lines indicate possibly non-simple iterations:

![Diagram](image)

**Claim 38.** \( k \) is \( \Sigma_0^{(n)} \) preserving.

**Proof.** By Lemma 11 the copy map \( j^* \) is \( \Sigma_0^{(n)} \) preserving and agrees with \( \bar{\sigma}_Y \) below the first coiteration index \( \nu_0 \). Since \( Q_0 \) is a common initial segment of \( \bar{M}_X, \bar{M}_Y \) then \( \nu_0 \geq \gamma_0 \), so

(17) \[ \bar{\sigma}_Y \upharpoonright Q_0 = j^* \upharpoonright Q_0 = \sigma_{0,Y} \]

First, we show that \( k[H^\omega_{\bar{N}^*_X} | \subseteq H^\omega_{\bar{S}^*} \big]. \) Let \( z^X_i \in H^\omega_{\bar{N}^*_X} \); then \( \bar{\pi}_X(z^X_i) \in H^\omega_{\bar{M}_X} \) and since \( \sigma_X \) is \( n \)-cofinal, then we can assume \( f^X_i \in H^\omega_{\bar{M}_X} \). Thus

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8This requirement is irrelevant if \[ \text{CASE (B)} \] holds.
\[ \tilde{f}_i^X \in H^n_{\tilde{M}_X} \] and so \( j^* \circ \tilde{\pi}_X(\tilde{f}_i^X) \in H^n_{\tilde{S}^X} \). Now \( \alpha_i^X < \sigma_X(\text{dom}(f_i^X)) = \sigma_Y \circ \sigma_{XY}(\text{dom}(f_i^X)) \), so:

(18) \[ \sigma_Y^{-1}(\alpha_i^X) < \sigma_{XY}(\text{dom}(f_i^X)) = \sigma_{XY} \circ \sigma_{0X}(\text{dom}(\tilde{f}_i^X)) = \sigma_{0Y}(\text{dom}(\tilde{f}_i^X)) = j^*(\text{dom}(\tilde{f}_i^X)) \]

(the last equality is due to [17]).

If [CASE (A)] holds, then as section 3.1 the \( \tilde{M}_X \) side of the coiteration is above \( \gamma_0 \) (see [9]) and \( \gamma_0 > \text{dom}(\tilde{f}_i^X) \); so in this case \( j^*(\text{dom}(\tilde{f}_i^X)) = j^* \circ \tilde{\pi}_X(\text{dom}(\tilde{f}_i^X)) \). Then by (18), \( \sigma_Y^{-1}(\alpha_i^X) \) is in the domain of \( j^* \circ \tilde{\pi}_X(\text{dom}(\tilde{f}_i^X)) \); and since this latter function is an element of the \( n \)-th reduct of \( \tilde{S}^* \), then \( k(z_i^X) = j^* \circ \tilde{\pi}_X(\text{dom}(\tilde{f}_i^X)) \) is an element of the \( n \)-th reduct of \( \tilde{S}^* \).

If [CASE (B)] holds, then recall \( \text{dom}(\tilde{f}_i^X) = \mu_0 \) for every \( i \in \omega \). Since the coiteration of \( \tilde{M}_X, \tilde{M}_X \) is above \( \mu_0 \) we have \( j^* \circ \tilde{\pi}_X(\text{dom}(\tilde{f}_i^X)) = j^* \circ \tilde{\pi}_X(\text{dom}(\tilde{f}_i^X)) \) (note \( \mu_0 < \text{cr}(j^*) \) by [18]). So again by (18), we have that \( \sigma_Y^{-1}(\alpha_i^X) \) is in the domain of \( j^* \circ \tilde{\pi}_X(\text{dom}(\tilde{f}_i^X)) \) and \( k(z_i^X) \) is an element of the \( n \)-th reduct of \( \tilde{S}^* \). This completes the proof that \( k[H^n_{\tilde{N}^*_X}] \subseteq H^n_{\tilde{S}^*} \).

Now suppose \( \Phi \) is a \( \Sigma_0^{(n)} \) formula, and let \( z_{i_1}^X, ..., z_{i_l}^X \) be elements of \( \tilde{N}_X^* \). Let \( f_i^* \) denote the function \( j^* \circ \tilde{\pi}_X(\tilde{f}_i^X) \). Then:

(19) \[ \tilde{N}_X^* \models \Phi(z_{i_1}^X, ..., z_{i_l}^X) \]

\[ \iff \tilde{N}_X \models \Phi(\pi_X(z_{i_1}^X), ..., \pi_X(z_{i_l}^X)) \]

\[ \iff \tilde{N}_X \models \Phi(\sigma_X(f_i^X)(\alpha_{i}^X), ..., \sigma_X(f_i^X)(\alpha_{i}^X)) \]

\[ \iff \alpha_{i_1}^X, ..., \alpha_{i_l}^X \models \sigma_{0X}(u_{\Phi,f_i^X}^{M_X}) \]

\[ \iff \sigma_Y^{-1}(\alpha_{i_1}^X), ..., \sigma_Y^{-1}(\alpha_{i_l}^X) \models \sigma_Y^{-1} \circ \sigma_X(u_{\Phi,f_i^X}^{M_X}) = \sigma_{XY}(u_{\Phi,f_i^X}^{M_X}) \]

\[ = \sigma_{XY}(u_{\Phi,f_i^X}^{M_X}) = \sigma_{0Y}(u_{\Phi,f_i^X}^{M_X}) = j^*(u_{\Phi,f_i^X}^{M_X}) \]

If [CASE (A)] holds, then \( \text{cr}(\tilde{\pi}_X) \geq \gamma_0 \) and \( \gamma_0 > \text{dom}(\tilde{f}_i^X) \) for every \( i \in \omega \), so \( j^*(u_{\Phi,f_i^X}^{M_X}) = j^* \circ \tilde{\pi}_X(\text{dom}(\tilde{f}_i^X)) = \text{cr}(\tilde{f}_i^X) \). Combined with (19), this implies that \( \tilde{N}_X^* \models \Phi(z_{i_1}^X, ..., z_{i_l}^X) \iff \tilde{S}^* \models \Phi(k(z_{i_1}^X), ..., k(z_{i_l}^X)) \).

If [CASE (B)] holds, then \( \text{cr}(\tilde{\pi}_X) \geq \mu_0 \) and \( \mu_0 = \text{dom}(\tilde{f}_i^X) \) for every \( i \in \omega \), so \( j^*(u_{\Phi,f_i^X}^{M_X}) = j^*(\mu_0 \cap u_{\Phi,f_i^X}^{M_X}) = j^*(\mu_0) \cap u_{\tilde{S}_f}^{\tilde{f}} = \mu_0 \cap u_{\tilde{S}_f}^{\tilde{f}} \).
Combined with (19), this implies that $\hat{N}_X \models \Phi(z_{i_1}^X, \ldots, z_{i_l}^X) \iff \hat{S} \models \Phi(k(z_{i_1}^X), \ldots, k(z_{i_l}^X))$.

Lemma 14 and Claim 38 imply $N_X$ is $n$-iterable, which is a contradiction. This completes the proof in section 3.2.2.

3.2.3. When $n = 0$. We are still assuming $L_X$ is a premouse; but now $\hat{P}_X$ might be a protomouse. As in section 3.1.4, the premouse $\hat{R}_X$ associated with $\hat{P}_X$ is wellfounded, and again we obtain Diagram 15. Since $n = 0$ then the functions $\hat{f}_i^X$ are elements of $\hat{P}_X$ and thus of $\hat{R}_X$. Similarly to section 3.2.2, $N_X^*$ can be $\Sigma_0$-embedded into the premouse $S^*$ at the end of Diagram 15, which contradicts the fact that $N_X^*$ is not $0$-iterable.

3.2.4. When $L_X$ is a protomouse. For this section we assume $L_X$ is a protomouse for some $X \in S$ (recall $L_X = ult^{n_X}(\hat{M}_X, \sigma_X)$ was defined at (16)). This implies that $n = 0$, since otherwise $\sigma_X$ would be $\Sigma_2$-preserving and the premousehood of $\hat{M}_X$ would carry upward to $L_X$.

Since $L_X$ is not a premouse—i.e. its top extender $F_X$ is not total—let $\delta_X$ be maximal such that $F_X$ is total on $L_X|\delta_X = Q|\delta_X$. Let $\tau_X := \mu^{|Q_X}|Q_X$. Then $\sigma_X$ does not map $\tau_X$ cofinally to $\sigma_X(\tau_X)$ (otherwise $F_X$ would be total on $L_X$). In particular, the continuity of $\sigma_X \upharpoonright Q_X$ on $\text{cof}(\omega)$ (see (2)) implies that $\tau_X$ and $\tilde{\tau}_X := \sup(\sigma_X[\tau_X]) = \mu^{|Q_X|\delta_X}$ have uncountable cofinality.

$\mu$ does not index an extender in $Q$ since $\mu$ is measurable in $Q$, and $\hat{Q}|\delta_X$ is sound above $\mu$. Then by Lemma 16, the premouse $R_X$ associated with $L_X$ (see Definition 25) is wellfounded and fully $m$-iterable, where $m := \text{degree}_{\hat{Q}|\delta_X}(\mu)$. This completes the proof of Lemma 31.
4. Consequences of the Frequent Extensions of Embeddings Lemma

Lemma 39 whose argument is due to Mitchell, is key to the proof of the well-known Weak Covering Lemma (Lemma 42) below. We note that once the Weak Covering Lemma has been proved, then Lemma 39 can be strengthened to obtain Lemma 44 in cases where $X \cap \text{Ord}$ is closed under limits of countable cofinality.

Lemma 39. Assume 0-pistol does not exist, and let $K$ be the core model. Let $S$ be a stationary subset of weakly internally approachable sets in $P_\kappa(H_\chi)$ where $\kappa \geq \omega_2$ and $\chi$ is a large regular cardinal. For $X \in S$ let $\pi_X : H_X \rightarrow X \prec (H_\chi, \in, K \cap H_\chi, \ldots)$ be the inverse of the transitive collapsing map, and $K_X := \pi_X^{-1}[K \cap H_\chi]$. Let $\alpha_X = X \cap \kappa = \text{cr}(\pi_X)$ and $\sigma_X := \pi_X \upharpoonright K_X$.

For the coiteration of $K_X$ with $K$, let

$$(M^X_i, \pi^M_i X, N^X_i, \pi^N_i X, \nu^X_i, \kappa^X_i, \tau^X_i, \delta^X_i, (N^X_i)^* | i \leq j \leq \theta_X)$$

denote the coiteration data as in Definition 9, where $K = N^X_0$ and $K_X = M^X_0$. Then for almost every $X \in S$:

1. $K_X$ does not move in the coiteration; i.e. there are no extenders applied on the $K_X$ side of the coiteration.
2. $\alpha^{+K_X}_X$ is not a cardinal in $K_X$\(^9\).
3. If $\nu^X_0$ is defined (i.e. $K_X$ is not an initial segment of $K$) then $\nu^X_0 < \alpha^{+K_X}_X$.
4. The $K$ side truncates by stage 1. There are two possibilities:
   CASE (A): $\tau^X_0 \geq \alpha_X$ and $K_X$ truncates at stage 0, or
   CASE (B): $\tau^X_0 < \alpha_X$ and $(\nu^X_0)^{+K_X}$ is not a cardinal in $K_1 := \text{ult}(K, E_{\nu^X_0})$.\(^10\)
5. Let $i^X_0 = 0$ if CASE (A) holds and $i^X_1 = 1$ if CASE (B) holds; so $i^X_0$ is the first truncation stage. Then for every $j \geq i^X_0$ the ultimate projectum of $(N^X_j)^*$ is $\leq \kappa^X_j$ and $(N^X_j)^*$ is sound above $\kappa^X_j$.

Here is a sketch of the proof; more details can be found in [10].

Proof. Suppose that $\alpha^{+K_X}_X$ were a cardinal in $K$ for stationarily many $X \in S$. For such $X$ it makes sense to form $\text{ult}(K, \sigma_X \upharpoonright \alpha^{+K_X}_X)$; and by Lemma 31 this ultrapower is wellfounded for almost every such $X$.

\(^9\)So by the convention of Definition 9 we define $\kappa^X_0, \tau^X_0, \delta^X_0$, and $(N^X_0)^*$ even if $K_X$ is an initial segment of $K$.

\(^10\)Then by the convention in Definition 9 the objects $\kappa^X_1, \tau^X_1, \delta^X_1$, and $(N^X_1)^*$ are defined even if $K_X$ is an initial segment of $K_1$ (if that happens here, then $\kappa^X_1 = \nu^X_0$).
Let $\bar{\sigma}_X : K \to K'_X$ denote the corresponding lifting. By Lemma 28, $\bar{\sigma}_X$ is a simple normal iteration map of $K$ with critical point $\alpha_X$; let $\nu$ be the first iteration index. So $\nu$ indexes an extender in $K$ but not in $K'_X$. By Lemma 7 and the normality of the iteration, $\nu < \bar{\sigma}_X(\alpha_X)$.

This is a contradiction, since $\bar{\sigma}_X(\alpha_X) = \sigma_X(\alpha_X) = \kappa$ and $K'_X$ end-extends $K|sup(\sigma_X[\alpha^+_{X,K}])$ (because $K'_X$ was defined as the ultrapower associated with the canonical lifting of $\sigma_X|\alpha^+_{X,K}$ to $K$).

Item (3) is a simple consequence of item (2) and coherency. If $\nu^X_0$ were $\geq \alpha^+_X$, then $\alpha^+_X = \alpha^+_X$ which contradicts item (2).

Now we prove (4). If $\tau_0^X = (\kappa^X)^+_{X,K} = \alpha_X$ then clearly $\tau_0^X$ is not a cardinal in $K$, since $\alpha_X = cr(\sigma_X)$. If $\tau_0^X > \alpha_X$ then (2) implies that $\tau_0^X$ is not a cardinal in $K$. For the remaining case $\alpha_X > \tau_0^X$ we use the following claim:

**Claim 40.** If $\alpha_X > \tau_0^X$ (so there is no truncation at stage 0), then for almost every $X \in S$:

- $\nu^X_0$ does not index an extender in $K_X$.
- $(\nu^X_0)^+_{X,K}$ is not a cardinal in $N^X_1 = \text{ult}(K, E^{K}_{\nu^X_0})$ (this implies that the $K$ side truncates at stage 1, since $\kappa^X_1 > \nu^X_0$).

**Proof.** Suppose that $\nu^X_0$ did index an extender in $K_X$; let $j_X$ be the ultrapower map for $ult(K_X, E^{K}_{\nu^X_0})$. Then since $\tau_0^X < \alpha_X$ is a cardinal in $K$, Lemma 31 guarantees that $ult(K, j_X \upharpoonright \tau_0^X)$ is wellfounded for almost every $X \in S$. It follows easily that $E^{K}_{\nu^X_0}$ is $K$-correct and is on $K$’s extender sequence by Lemma 28, contradicting the fact that $\nu^X_0$ is a coiteration index.

To see that $(\nu^X_0)^+_{X,K}$ is not a cardinal in $N^X_1$ for almost every $X \in S$, suppose it were. By the first part of this claim and the coherency of the extender $E^{K}_{\nu^X_0}$, then $E^{K}_{\nu^X_0} = \emptyset$ and so $K_X|((\nu^X_0)^+_{X,K})$ is an initial segment of $N^X_1$. Using Lemma 31 lift $\pi_X \upharpoonright (\nu^X_0)^+_{X,K}$ to $N^X_1$ and let $\tilde{\pi}_X : N^X_1 \to L^X$ be this lifting. Then $\tilde{\pi}_X \circ \pi_{0,1}^X$ is a map from $K \to N^X_1 \to L_X$, where $\pi_{0,1}^X$ is the ultrapower map for the first step of the $K$ side (i.e. $\pi_{0,1}^X : K \to E^{K}_{\nu^X_0}N^X_1$). By Lemma 28 the map $\tilde{\pi}_X \circ \pi_{0,1}^X$ must be a simple normal iteration map, and the first part of it—to $N^X_1$—used the index $\nu^X_0$. So normality of this iteration requires that all critical points in the part from $N^X_1$ to $L_X$ must be strictly above $\nu^X_0$. But the critical point of $\tilde{\pi}_X$ is $\alpha_X$, and $\alpha_X \leq \nu^X_0$. This is a contradiction. □
Item 5 then follows by a straightforward induction on \( \iota \geq \iota^X_0 \); this is a general fact about a normal linear iteration after a truncation has occurred.

Finally we prove (1). Suppose this fails, and let \( \iota_X \) be the first stage where the \( K_X \) side applies an extender. By Claim 41 we can assume that \( \iota_X \geq \iota^X_0 \). By applying the Fodor Lemma for \( P_\kappa(H_\theta) \), we can WLOG assume that \( \pi_X(\nu^X_{\iota X}) \) is the same for every \( X \in S \); let \( \nu' \) denote this common value. Let \( j : K \rightarrow E^\kappa_X \) and let \( \kappa' = cr(E^\kappa_X) \).

For each \( X \) let:
- \( Q_X := K_X |_{\tau^X_{\iota X}} N_X \)
- \( \sigma_X := j \circ \pi_X \upharpoonright Q_X \).
- \( \bar{F}_X := E^\kappa_X, \bar{K}_X := cr(\bar{F}_X), \) and \( \bar{\tau}_X = \tau^X_{\iota X} = \bar{K}_X^{+K_X} \)
- \( N^*_X := (N^*_X)^* \)
- \( n_X := deg_{N^*_X}(\kappa^X_{\iota X}); \) WLOG we will assume this has a fixed value \( n \) for every \( X \in S \).

Since there is an extender on \( \kappa^X_{\iota X} \) in \( K_X \), then Lemma 7 implies that \( o^{K_X}(\mu) = o^{N^*_X}(\mu) < \kappa^X_{\iota X} \) for every \( \mu < \kappa^X_{\iota X} \). Then the canonical lifting of \( \sigma_X \) guaranteed by Lemma 31 yields a premouse which end-extends \( K[j(\kappa')] \) and is \( n \)-iterable; the fact that this lifting yields a premouse (and not a protomouse) follows from Lemma 7. Let \( \bar{\sigma}_X : \bar{N}^*_X \rightarrow \bar{N}^\prime_X \) be this canonical lifting. Also, since \( \bar{N}^*_X \) is sound above \( \kappa^X_{\iota X} \) then \( \bar{N}^*_X \) is sound above \( j(\kappa) = \bar{\sigma}_X(\bar{K}_X) \) and \( \omega^{n+1} \leq j(\kappa') \).

\( N^*_X \) is in fact fully iterable: \( N^*_X \) and \( K' \) agree below \( j(\kappa') \) and \( j(\kappa') \) does not index an extender in either, so their attempted \( * \)-coiteration is above \( j(\kappa') \gtrsim \omega^{n+1} \). Thus the \( n \)-iterable premouse \( N^*_X \) is \( * \)-coiterable with \( K' \) and so \( N^*_X \) is fully iterable (since \( K' \) is universal and thus the \( N^*_X \) side of the coiteration is simple).

Now \( \bar{F}_X \) is on the sequence of the mouse \( K_X \) and so is weakly amenable with respect to \( K_X | \bar{\tau}_X = N^*_X | \bar{\tau}_X \). Define a map \( k : ult^*(\bar{N}^*_X, \bar{F}_X) \rightarrow \bar{N}^*_X \) by \( [\beta, f]_{\bar{F}_X} \rightarrow \bar{\sigma}_X(f)(\pi_X(\beta)) \) where \( \beta < lh(\bar{F}_X) = \nu^X_{\iota X} \) and \( f \) is a good \( \Sigma^a_{1(n-1)}(\bar{N}^*_X) \) function. It is straightforward to check that that \( k \) is well-defined, \( \Sigma^a_0 \) preserving, and \( n \)-cofinal. Let \( \bar{N}^*_X := ult^*(\bar{N}^*_X, \bar{F}_X) \).

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11 Alternatively, one could first lift \( N^*_X \) to an initial segment of \( K \), take the ultrapower of that lifting by \( E_{\nu'} \), and then embed \( \bar{N}_X \) into that ultrapower; this amounts to the same thing.

12 Recall \( \pi_X \) is the map from \( K_X \rightarrow K \) and \( \bar{\sigma}_X \) extends \( j \circ \pi_X \upharpoonright \bar{\tau}_X \).
and \( \pi_{F_X} \) be the corresponding ultrapower map. This map is \( \Sigma_0^{(n)} \) preserving and \( n \)-cofinal and \( \pi_{F_X}(p) \in R_{\tilde{N}_X}^n \) where \( p \) is the standard parameter of \( \tilde{N}_X \) above the \( n \)-th projectum. Lemma [4] guarantees that \( \tilde{N}_X \) is \( n \)-iterable and that in the \( n \)-coiteration of \( \tilde{N}_X \) with \( \tilde{N}_X \), the \( N^*_X \) side is simple. Also, it is easy to check that the critical points of this coiteration are \( \geq \bar{\kappa}_X \) which in turn is \( \geq (n+1) \)st projecta of both \( N^*_X \) and \( \tilde{N}_X \); so this \( n \)-coiteration is really just the \( * \)-coiteration.

**Claim 41.** \( N^*_X \) and \( \tilde{N}_X \) coiterate simply to a common mouse.

**Proof.** We have already seen that the \( N^*_X \) side is simple. Since \( \pi_{F_X} \) can be viewed as a lifting of the ultrapower of \( K_X | \bar{\tau}_X \) by \( \bar{F}_X \), then:

\[
\begin{align*}
20) & \tilde{N}_X \text{ agrees with } M_{i_X+1}^{X} \text{ below } \pi_{F_X}(\kappa_{i_X}^X) \\
21) & \nu_{i_X}^X \text{ does not index an extender in } \tilde{N}_X \text{ (by the coherency of } \bar{F}_X \text{).}
\end{align*}
\]

CASE 1: \( \nu_{i_X} \) indexes an extender in \( N^*_X \).
In this case the \( N^*_X \) side applies \( E_{\nu_{i_X}^X}^{N^*_X} \) at the first step of the coiteration; let \( N'^* \) denote the resulting mouse. Then:

\[
\begin{align*}
(1) & N'^* \text{ and } \tilde{N}_X \text{ coiterate above } \nu_{i_X}^X \\
(2) & N'^* \text{ and } \tilde{N}_X \text{ have the same successor of } \nu_{i_X}^X \\
(3) & \text{The expansions of } N'^* \text{ and } \tilde{N}_X \text{ are both sound above } \nu_{i_X}^X
\end{align*}
\]

Item [2] is due to the fact that the \( K_X \) side of the \( K \) vs. \( K_X \) coiteration is simple, while at the same time the \( N^*_X \) side of the \( N^*_X \) vs. \( \tilde{N}_X \) coiteration is simple.

Standard arguments then show that \( N'^* \) and \( \tilde{N}_X \) coiterate simply to a common mouse, and that in fact \( N'^* = \tilde{N}_X \).

CASE 2: \( \nu_{i_X} \) does not index an extender in \( N^*_X \).
In this case, both sides of the coiteration \( N^*_X \) vs. \( \tilde{N}_X \) are above \( \nu_{i_X}^X \). Similarly to Case 1, \( N^*_X \) and \( \tilde{N}_X \) have the same successor of \( \nu_{i_X}^X \), and the argument is similar to Case 1. \( \square \)

Let \( R \) be the common simple iterate from Claim [4], \( k : \tilde{N}_X \rightarrow R \) be the iteration map on the \( N^*_X \) side, and \( \bar{k} : \tilde{N}_X \rightarrow \hat{R} \) be the iteration map on the \( \tilde{N}_X \) side. Note that \( cr(\bar{k}) > \bar{\kappa} \) by (20) and (21). The soundness of \( \tilde{N}_X \) above \( \kappa_{i_X}^X \), along with the preservation degree of \( \pi_{F_X}, \bar{k}, \) and \( k \), guarantees that \( k \circ \pi_{F_X} = k \); so in fact \( \nu_{i_X}^X \) must index an extender in \( N^*_X \) (i.e. Case 2 from Claim [4] is impossible). The commutivity of the diagram, and the fact that \( cr(\bar{k}) > \kappa_{i_X}^X \), imply that for every \( \eta < \nu_{i_X}^X \)
and \( z \in K_X | \tau_X \):

\[
\eta \in \bar{F}_X(z) \iff \eta \in E^{N_X}_{\nu_X}(z)
\]

In other words, \( E^{N_X}_{\nu_X} \) and \( \bar{F}_X \) are the same extender. This is a contradiction, since \( \nu_X \) was a coiteration index. \( \square \)

The following two lemmas are proved by a joint induction; the induction assumption guarantees that requirement (1) of Lemma 31 is satisfied when \( Q \) is an initial segment of \( K \).

**Lemma 42.** (Weak Covering Lemma) Assume 0-pistol does not exist, and let \( K \) be the core model. If \( \kappa \geq \omega_2 \) then \( \text{cf}(\kappa^{+K}) \geq |\kappa| \).

**Proof.** The argument relies heavily on Lemma 39. A detailed proof of Weak Covering for the core model for measures of order zero appears at Theorem 7.5.1 in [10]; on page 278 he points out the minor revisions needed to prove Weak Covering for the core model below 0-pistol. \( \square \)

**Lemma 43.** Consider the special case of Lemma 31 where \( Q \) is an initial segment of the core model \( K \), \( \gamma \) is a cardinal in \( K \), \( n_X = \text{degree}_{\bar{M}_X}(\gamma_X) \), and \( \bar{M}_X \) is sound above \( \gamma_X \). Then \( L_X = \text{ult}(\bar{M}_X, \sigma_X) = \text{ult}^{n_X}(\bar{M}_X, \sigma_X) \) is wellfounded.

If \( L_X \) is a premouse, then letting \( N_X \) be its associated bookkeeping premouse (i.e. \( L_X = \bar{N}_X \)):

- If \( o^K(\mu) < \gamma \) for every \( \mu < \gamma \), then \( N_X \) is an initial segment of \( K \)
- If there is some \( \mu < \gamma \) such that \( \gamma \leq o^K(\mu) \), then \( o^{N_X}(\mu) \leq o^K(\mu) \). If \( o^{N_X}(\mu) = o^K(\mu) \), then \( N_X \) is an initial segment of \( K \). If \( o^{N_X}(\mu) < o^K(\mu) \) then \( N_X \) is an initial segment of \( \text{ult}(K, E^K_{o^K(\mu)}) \).

If \( L_X \) is not a premouse, let \( R_X \) be the premouse associated with \( L_X \) (see Definition 25). Let \( F_X \) be the non-weakly-amenable top extender of \( L_X \), \( \mu = \text{cr}(F_X) \), and \( \lambda_X = \text{lh}(F_X) \) ( = the largest cardinal in \( L_X \)). Then \( o^K(\mu) \geq o^{R_X}(\mu) \), and:

- If \( o^K(\mu) = o^{R_X}(\mu) \) then \( R_X \) is an initial segment of \( K \)
- If \( o^K(\mu) > o^{R_X}(\mu) \) then \( R_X \) is an initial segment of \( \text{ult}(K, E^K_{o^{R_X}(\mu)}) \).

**Proof.** Suppose first that \( L_X \) is a premouse and \( N_X \) is its cutback. The soundness of \( \bar{M}_X \) above \( \gamma_X \) implies that \( L_X \) is sound above \( \gamma \) (and that \( \omega^{p^{N_X}_{L_X+1}} \leq \gamma \)). If \( \gamma \) does not index an extender in \( K \), or it indexes an extender on some \( \mu < \gamma \) and \( o^{N_X}(\mu) = o^K(\mu) \), then \( N_X \) and \( K \) coiterate above \( \gamma \). It follows that \( N_X \) is an initial segment of \( K \).
If $o^{N_X}(\mu) < o^K(\mu)$, then at the first stage the $K$ side applies the extender $E^K_{o^{N_X}(\mu)}$; let $K_1 = ult(K, E^K_{o^{N_X}(\mu)})$. Then $K_1$ and $N_X$ coiterate above $\gamma$, and $N_X$ is an initial segment of $K_1$.

We can rule out the case that $o^{N_X}(\mu) > o^K(\mu)$: the inductive assumption that (1) of Lemma 31 holds in $Q$, along with Corollary 19, implies that any extender on $\mu$ in $N_X$ would be $K$-correct. Since $K$ absorbs $K$-correct extenders (see Lemma 28), then such an extender would be on $K$’s sequence.

Now suppose that $L_X$ is not a premouse; i.e. its top predicate $F_X$ is not weakly amenable. As in section 3.2.4, the premouse $\hat{R}_X$ associated with $L_X$ (see Definition 25) exists and is fully $m_X$-iterable. Let $\pi : \tilde{L}_X|\delta \rightarrow^* \tilde{R}_X$. By Lemma 26, $\hat{R}_X$ is sound above the sup of generators of $F_X$, which is in turn $\leq o^{R_X}(\mu)$. As in the case where $L_X$ was a premouse, we can rule out the case that $o^{R_X}(\mu) > o^K(\mu)$. So in the coiteration of $R_X$ with $K$, the $R_X$ side is above $o^{R_X}(\mu)$. The rest of the argument is similar to the case when $L_X$ was a premouse, so $R_X$ is either an initial segment of $K$ or of $K_1$. Finally, we note that the protomouse $L_X$ is computable from $\hat{R}_X$ (see Lemma 26), so $L_X \in K$ (in fact an element of $K|\gamma^K$, since $L_X$ is sound above $\gamma$). □

The following is a different version of Lemma 39. Its proof is similar to the proof of Lemma 39 except that Lemmas 16, 17, and 42 are used instead of the Frequent Extensions of Embeddings Lemma:

**Lemma 44.** Assume $X \prec (H_\theta, \in, ...) \text{ and } X \cap \text{Ord is } \omega\text{-closed}$. Let $\pi_X : H_X \rightarrow X$ be the inverse of the transitive collapse map, $\alpha = cr(\pi_X)$, and $K_X$ the core model constructed in $H_X$. Assume $\sigma_X(\alpha_X) \geq \omega_2$. Then the conclusions of Lemma 39 hold for the $K$ vs. $K_X$ coiteration.

Note that we do not place any cardinality restrictions on $X$. For example, $X$ might be an $\omega_2$-sized structure witnessing $(\omega_3, \omega_2) \rightarrow (\omega_2, \omega_1)$. 
5. Covering Theorems

In this section we prove Theorems 45 and Theorem 46.

**Theorem 45.** Assume \( \gamma > \omega_2 \), \( cf(\gamma) < |\gamma| \), and \( \gamma \) is regular in \( K \). Then \( \gamma \) is measurable in \( K \).

If \( cf(\gamma) \) is uncountable, the consequence is stronger:

**Theorem 46.** Assume \( \gamma > \omega_2 \), \( \omega < cf(\gamma) < |\gamma| \), and \( \gamma \) is regular in \( K \). Then:

1. For every \( \beta < cf(\gamma) \), \( \{ \lambda < \gamma | o^K(\lambda) \geq \beta \} \) contains a closed unbounded set in \( \gamma \),
2. \( o^K(\gamma) \geq cf(\gamma) \).

Part 2 of Theorem 46 is really a consequence of part 1 and the following theorem:

**Theorem 47.** Assume \( \gamma > \omega_2 \), \( \omega < cf(\gamma) < |\gamma| \), and \( \gamma \) is regular in \( K \). Let \( a \in P^K(\gamma) \) and let \( S_a \) be the collection of \( \lambda < \gamma \) such that \( K \) has a normal measure on \( \lambda \) which concentrates on \( a \cap \lambda \). Assume \( S_a \) is stationary in \( V \). Then \( K \) has a normal measure on \( \gamma \) which concentrates on \( a \).

Theorem 47 will also be used in section 6.

We note the following, although it is not used in the proof of any of the theorems above:

**Note 48.** If \( \gamma \) is as in the hypothesis of Theorem 45 then

1. \( \gamma \) is inaccessible in \( K \)
2. either:
   - \( \gamma \) is a singular cardinal in \( V \), or
   - \( \gamma \in (\kappa, \kappa^+) \) for some \( \kappa \) which is regular in \( V \).

**Proof.** To see that \( \gamma \) is inaccessible in \( K \), suppose otherwise; say \( \gamma = \lambda^K \). Then if \( \gamma \) is not a cardinal in \( V \), by Lemma 42 (the weak covering lemma) \( cf(\gamma) \geq |\lambda| = |\gamma| \), contrary to our assumption. If \( \gamma = \lambda^K \) is also a cardinal in \( V \) then it must be a successor cardinal in \( V \), again contrary to our assumption that \( \gamma \) is singular in \( V \).

To see why (2) must hold: if \( \gamma \) is not a cardinal in \( V \), then \( |\gamma| \) must be regular in \( V \), since \( K \) correctly computes the successors of singular cardinals in \( V \) by Lemma 42 (the weak covering lemma). \( \square \)

---

13Recall from Definition 6 that for mice below 0-pistol, \( o^M(\kappa) \) denotes the “Mitchell order of \( \kappa \) in \( M \)”; more precisely, the ordertype of the collection of \( \nu \geq \kappa^+M \) which index an extender with critical point \( \kappa \). On the other hand \( o^M(\kappa) \) denotes the least primitively recursively closed ordinal which does not index an extender with critical point \( \kappa \).
Theorems 45, 46, and 47 are proved in sections 5.1, 5.2, and 5.3, respectively.

5.1. Proof of Theorem 45. In this section we prove Theorem 45. Let \( \kappa := \max(cf(\gamma)^+, \omega_2) \), and pick a large regular \( \chi >> \gamma^+ \). Let \( S \) be the set of all weakly internally approachable \( X \) of cofinality \( \chi \) by a standard closure argument.

By Lemma 39 (i.e. \( (even if \( K \) segment of \( \gamma \) projects strictly below \( \hat{\gamma} \)), and pick a large regular \( \chi \)). For almost every \( X \in S \) we have:

1. \( \alpha_\chi^X \) is not a cardinal in \( K \). Then by the convention in Definition 9 if \( K_X \) is an initial segment of \( K \) then \( \kappa_0^X \) is defined to be \( \alpha_\chi, \tau_0^X = \alpha_\chi^X \) is maximal such that \( \tau_0^X \) is a cardinal in \( K \), and \( (N_0^X)^* = K \) if \( \kappa_0^X \) is defined.

2. If \( \nu_0^X \) is defined (i.e. \( K_X \) is not an initial segment of \( K \)) then \( \nu_0^X < \alpha_\chi^X \).

3. The \( K \) side truncates by stage 1. By Lemma 39 there are only two possibilities:
   - CASE 0: \( \tau_0^X \geq \alpha_X \) and \( K \) truncates at stage 0, or
   - CASE 1: \( \tau_0^X < \alpha_X \) and \( (\nu_0^X)^{+K_X} \) is not a cardinal in \( K_1 := \text{ult}(K, E_{\nu_0^X}) \). By the convention in Definition 9, the objects \( \kappa_1^X, \tau_1^X, \delta_1^X \), and \( (N_1^X)^* \) are defined even if \( K_X \) is an initial segment of \( K_1 \) (if that happens here, then \( \kappa_1^X = \nu_0^X \)).

4. The \( K_X \) side of the coiteration is trivial; note that this is equivalent to \( \nu_i^X = \delta^{K_X}(\kappa_i^X) \) whenever \( \nu_i^X \) is defined.

5. Let \( \iota_X = 0 \) if Case 0 holds and \( \iota_X = 1 \) if Case 1 holds (i.e. \( \iota_X \) is the stage of the first truncation). Then for every \( i \geq \iota_X \):
   - the ultimate projectum of \( \widehat{(N_i^X)}^* \) is \( \leq \kappa_i^X \) and \( (N_i^X)^* \) is sound above \( \kappa_i^X \).

So if Case 0 holds then \( (N_0^X)^* \) is defined (even if \( K_X \) is an initial segment of \( K \)) and \( (N_0^X)^* \) projects to or below \( \alpha_X \); in particular it projects strictly below \( \gamma_X \). If Case 1 holds, then \( (N_1^X)^* \) is defined (even if \( K_X \) is an initial segment of \( K_1 \)) and \( (N_1^X)^* \) projects to \( \nu_0^X \); the next claim will then imply that \( (N_1^X)^* \) projects strictly below \( \gamma_X \):

Claim 49. If Case 1 holds, then \( \nu_0^X < \gamma_X \).

\[ \text{For the purposes of Theorem 45, } \chi > \gamma^+ \text{ suffices.} \]
Proof. Suppose to the contrary that $\nu_0^X \geq \gamma_X$. We already know by item 2 in the list above that $\nu_0^X < \alpha^+_K$, so $\gamma_X \in (\alpha_X, \alpha^+_K)$; in particular $\gamma_X$ is singular in $K$. Let $\delta_X$ be maximal such that $\gamma_X$ is regular in $\hat{K}||\delta_X$. Then the ultimate projectum of $\hat{K}||\delta_X$ is $\leq \gamma_X$ and $\gamma_X$ is definably singularized over $\hat{K}||\delta_X$. Since $\nu_0^X \geq \gamma_X$, then $K_X|\gamma_X$ is an initial segment of $K$, so it makes sense to form the canonical lifting of $\sigma_X \upharpoonright \gamma_X$ to $\hat{K}||\delta_X$. By Lemma 43, this canonical lifting is wellfounded for almost all $X \in S$; let $\tilde{\sigma}_X : \hat{K}||\delta_X \rightarrow P_X$ be one such lifting. Now $\gamma_X$ is regular in $\hat{K}||\delta_X$, so $\tilde{\sigma}_X(\gamma_X) = sup_{\sigma_X}[\gamma_X] = \gamma$; and since $\gamma_X$ is definably singular over $\hat{K}||\delta_X$, then $\gamma$ is definably singular over $P_X$. But $P_X \in K$ by Lemma 43, so $\gamma$ is singular in $K$. Contradiction. $\Box$

To summarize:

\begin{align*}
(23) \quad (\hat{N}_0^X)^* \text{ is sound and projects strictly below } \gamma_X \text{ in Case 0} \\
(24) \quad (\hat{N}_1^X)^* \text{ is sound and projects strictly below } \gamma_X \text{ in Case 1}
\end{align*}

Since the $K_X$ side of the coiteration is trivial and the last iterate on the $K$ side must end-extend $K_X$, there is some minimal stage $\Omega_X$ such that the extender sequences of $\hat{N}_{\Omega_X}^X$ and $K_X$ agree below $\gamma_X$. Define:

\begin{align*}
(24) \quad N_X := \begin{cases} 
(\hat{N}_0^X)^* & \text{if } \Omega_X = 0 \\
(\hat{N}_1^X)^* & \text{if } \Omega_X = 1 \\
N_{\Omega_X}^X & \text{if } \Omega_X > 1
\end{cases}
\end{align*}

The height of $N_X$ is at least $\gamma_X$. Furthermore, since there may be a truncation at state $\Omega_X$, if $\Omega_X > 1$ then:

\begin{align*}
(25) \quad \text{If } \Omega_X > 1 \text{ then } N_X \text{ is not necessarily the same as } (\hat{N}_{\Omega_X}^X)^*.
\end{align*}

If $\Omega_X > 0$ (i.e. $K_X|\gamma_X$ is not an initial segment of $K$) then $J_X := \{j| j \geq \gamma_X \text{ and } \nu_j^X < \gamma_X\}$ is nonempty and $\Omega_X$ is the strict supremum of $J_X$. We will later see that the critical points of the coiteration are unbounded in $\gamma_X$; so the “strict” here will be superfluous and $\Omega_X$ will never be 0 or 1. Furthermore, using (23) it is easy to show:

\begin{align*}
(26) \quad \text{The ultimate projectum of } \hat{N}_X \text{ is } \leq sup(\{\nu_j^X| j \in J_X\}) \text{ and } \hat{N}_X \text{ is sound above } sup(\{\nu_j^X| j \in J_X\})
\end{align*}

**Lemma 50.** For almost every $X \in S$: $\gamma_X$ is not $\Sigma^*$ singularized over $\hat{N}_X$.

**Proof.** Suppose for a contradiction that $\gamma_X$ were definably singularized over $\hat{N}_X$ for stationarily many $X \in S$. For such $X$ pick $\eta_X$ maximal such that $\gamma_X$ is regular in $\hat{N}_X||\eta_X$. Then the ultimate projectum of
\( \widehat{N_X} \| \eta_X \) is \( \leq \gamma_X \) and \( \widehat{N_X} \| \eta_X \) is sound above \( \gamma_X \); in case \( \eta_X = ht(N_X) \) (i.e. \( \gamma_X \) is regular in \( N_X \)) this is guaranteed by \( (26) \). Then the Frequent Extensions of Embeddings Lemma yields liftings \( \bar{\sigma}_X : \widehat{N_X} \| \eta_X \to P_X \), where \( \gamma = \bar{\sigma}_X(\gamma_X) \) is regular in, but definably singularized over, the structure \( P_X \). By Lemma \( 43 \) \( P_X \in K \) and thus \( \gamma \) is singular in \( K \), a contradiction. \( \square \)

**Lemma 51.** For almost every \( X \in S \), \( D_X := \{ \kappa^X_i | \pi_{i, \Omega_X}(\kappa^X_i) = \gamma_X \} \) is a closed unbounded subset of \( \gamma_X \). (Note that we do not assume that \( \gamma \) has uncountable cofinality; the following proof would be a bit simpler if that were the case).

**Proof.** First, we note that

\[ (27) \text{ The critical points of the coiteration are cofinal in } \gamma_X \text{ for almost every } X \in S \]

Equivalently, \( \{ \nu^X_i | i \in J_X \} \) is cofinal in \( \gamma_X \). Suppose not; let \( s_X := sup \{ \nu^X_i | i \in J_X \} = sup \{ \nu^X_i | i < \Omega_X \} \). By \( (26) \), \( \widehat{N_X} \) is sound above \( s_X \); but then \( \gamma_X \) would be \( \Sigma^*(N_X) \)-singularized, contradicting Lemma \( 50 \).

Now we show \( D_X \) is unbounded for almost all \( X \in S \); closure then follows by basic properties of directed systems. Suppose for a contradiction that \( D_X \) is bounded for stationarily many \( X \in S \). We will show this implies \( \gamma_X \) is definably singularized over \( \widehat{N_X} \), which again contradicts Lemma \( 50 \). For the rest of the proof we omit the \( X \) subscripts and superscripts when there is no danger of confusion. Since \( D_X \) is bounded below \( \gamma_X \) there is some \( i^* < \Omega_X \) such that for every \( j \in [i^*, \Omega_X) \), \( \gamma_j := \pi_{i^* \Omega_X}(\gamma_X) \) is strictly greater than \( \kappa_j \). Also demand that the degree of the critical points (on the \( K \) side) past stage \( i^* \) is fixed at some \( n \in \omega \). This can be arranged since by \( (27) \), \( \Omega_X \) is a limit ordinal and the degrees of the critical points of a normal iteration constitute a nonincreasing sequence of natural numbers. Then \( (27) \) also guarantees:

\[ (28) \omega \rho^{p_{N_X}} \leq \gamma_X \leq \omega \rho^p_{N_X} \]

The following claim shows that \( \gamma_X \) is definably singularized over \( \widehat{N_X} \), thus contradicting Lemma \( 50 \) and completing the proof of Lemma \( 51 \).

**Claim 52.** Let \( h^{n+1}_{j} = h^{n+1}_{N_j}(-, p_{N_j}) \). Then \( \gamma_X \cap \pi_{i^* \Omega_X} \circ h^{n+1}_{i^*} [\kappa_{i^*}] \) is a cofinal subset of \( \gamma_X \). Note that the relation \( \pi_{i^* \Omega_X} \circ h^{n+1}_{i^*} [\kappa_{i^*}] \) is \( \Sigma^1_1(N_X) \); so the this claim shows \( \gamma_X \) is definably singularized over \( \widehat{N_X} \).
Proof of Claim 52. Recall that $\hat{N}_i^X$ is sound above $\kappa_i$; so we just need to show that $\pi_i^\ast \Omega_X$ is continuous at $\gamma_i^\ast$. This, in turn, requires showing that for every $j \in [i^\ast, \Omega_X)$, the map $\pi_{i,j+1}$ is continuous at $\gamma_j$. By (28), either every such $\gamma_j$ equals $\omega \rho_{\hat{N}_j}^n$, or else every $\gamma_j$ is an element of the $n$-th reduct of $\hat{N}_j$. In the former case, continuity of the iteration maps at each $\gamma_j$ follows from the closeness of the applied extenders and the fact that $n = degree \hat{N}_j (\kappa_j)$. In the latter case, notice first that since Lemma 50 guarantees that $\gamma_X$ is regular in $\hat{N}_X$, then by elementarity of $\pi_{i,X}$ we have $\gamma_j$ is regular in $\hat{N}_j$; in particular, $cf \hat{N}_j (\kappa_j) > \kappa_j$. Since $\gamma_j$ is an element of the $n$-th reduct and $n = degree \hat{N}_j^X (\kappa_j)$, then the $\ast$-ultrapower at stage $j$ agrees with the coarse ultrapower on the $n$-th reduct of $\hat{N}_j$, and is thus continuous at $\gamma_j$ since $cf \hat{H}_n (\gamma_j) > \kappa_j$. □ (Claim 52)

□ (Lemma 51)

Without loss of generality, for the rest of this section assume the statement of Lemma 51 holds for every $X \in S$. Then in particular every $\Omega_X$ is a limit ordinal, so for each $X$ there is an $n_X < \omega$ past which the degrees of the critical points stabilize; i.e., there is some $i_X^\ast < \Omega_X$ such that for every $j \in [i_X^\ast, \Omega_X)$, $n_X$ is maximal such that $\kappa_j^X$ is an element of the $n$-th reduct of $\hat{N}_j^X$. Without loss of generality assume $n_X$ is the same for every $X \in S$, say $n_X = n$.

Also, $\gamma_X$ is measurable in $\hat{N}_X$, since it is the image of measurables via the iteration maps. This implies:

Claim 53. $\gamma$ does not index an extender in $K$

Proof. Suppose it did; say $\gamma$ indexes an extender on some $\mu < \gamma$ in $K$. Then $\gamma_X$ indexes an extender on $\mu_X := \sigma_X^{-1} (\mu)$ in $K_X$. Since $K_X$ and $N_X$ agree below $\gamma_X$, this implies that $\sigma_X (\mu_X) \geq \gamma_X$, so that there are cofinally many extenders below $\gamma_X$ which index extenders on $\mu_X$ in $N_X$; this contradicts Lemma 7, since $\gamma_X$ is measurable in $N_X$ (recall we are assuming 0-pistol does not exist; i.e. no mice have overlapping extenders).

□

So the top extender of $N_X$ (if it has one) has critical point $\geq \gamma_X$; let $\mu_X$ denote this critical point. Then by Lemma 7, $\mathcal{U}^{\ast} (\hat{N}_X, \sigma_X \upharpoonright \gamma_X)$ a premouse (i.e. its top extender, if it has one, is weakly amenable), which we’ll denote by $N'_X$, and is in fact an initial segment of $K$ by Lemma 43.
The map \(\tilde{\sigma}_X\) is \(\Sigma_0^{(n)}\)-preserving and \(n\)-cofinal, since each \(\widehat{N}_X\) is sound above \(\gamma_X\). Let \(\tau_X\) denote the successor of \(\gamma_X\) in \(\widehat{N}_X\). Let \(D'_X := \sigma_X[D_X]\) and \(U'_X = \sigma_X(U_X)\), where \(U_X := U^{N_X}(\gamma_X, 0)\), i.e., the measure on \(\gamma_X\) of order zero in \(\widehat{N}_X\) (recall the convention [8]). Here \(\tilde{\sigma}_X(U_X)\) refers to the top predicate of \(\widehat{N}_X\), if \(U_X\) is the top predicate of \(\widehat{N}_X\).

For every pair \(X \subset Y\) in \(S\), let \(\sigma_{XY} = \sigma_Y^{-1} \circ \sigma_X \upharpoonright \gamma_X\). Using an interpolation-like argument with the map \(\tilde{\sigma}_X\), and the fact that \(\widehat{N}_X\) is sound above \(\gamma_X\), construct the following maps:

- \(\tilde{\sigma}_{XY} : \widehat{N}_X \rightarrow \varsigma_0^{(n)} \widehat{N}_{XY}\) which is \(n\)-cofinal and extends \(\sigma_{XY} \upharpoonright \gamma_X\); the ultrapower really is a premouse by Lemma 7, since \(\gamma_X\) does not index an extender in \(\widehat{N}_X\).
- \(\sigma'_{XY} : N^{XY} \rightarrow \varsigma_0^{(n)} \widehat{N}_X\) which extends \(\sigma_Y \upharpoonright \gamma_Y\)\(^\text{15}\).

The following diagram depicts the situation; the “hats” are omitted.

\[
\begin{array}{ccc}
N'_X & \xrightarrow{\sigma'_{XY}} & N_{XY} \\
\downarrow{\tilde{\sigma}}_X & \quad \quad \quad & \quad \quad \quad \downarrow{\tilde{\sigma}}_{XY} \\
N_X & \xrightarrow{\sigma_{XY}} & \widehat{N}_X \\
\end{array}
\]

Also note that \(\tilde{\sigma}_X\) and \(\tilde{\sigma}_{XY}\) are both continuous at \(\gamma_X^{+\widehat{N}_X}\), since they are \(n\)-cofinal and \(\gamma_X < \omega^{\rho_{\widehat{N}_X}}\).

**Claim 54.** \(N_{XY}\) is an initial segment of \(N_Y\).

**Proof.** The expansions of both \(N_{XY}\) and \(N_Y\) are sound above \(\gamma_Y\), and the preservation degree of \(\sigma'_{XY}\) guarantees that \(N_{XY}\) is normally iterable above \(\gamma_Y\). Finally, \(N_{XY}\) and \(N_Y\) both end-extend \(K_Y|\gamma_Y\) and coiterate above \(\gamma_Y\), since by claim (53) \(\gamma_Y\) does not index an extender in either \(N_{XY}\) or \(N_Y\). It follows that one of \(N_{XY}, N_Y\) is an initial segment of the other.

If \(N_Y\) were a proper initial segment of \(N_{XY}\), then \(\gamma_Y^{+\widehat{N}_Y} < \gamma_Y^{+\widehat{N}_{XY}}\) and \(\sigma'_{XY} \upharpoonright \widehat{N}_Y : \widehat{N}_Y \rightarrow \sigma'_{XY}(\widehat{N}_Y)\) would be a fully elementary map which extends \(\sigma_Y \upharpoonright \gamma_Y\). Using the interpolation lemma, it can be shown that \(\sigma'_{XY}\) and \(\tilde{\sigma}_Y\) would agree below \(\gamma_Y^{+\widehat{N}_Y}\) (their restrictions to \(\gamma_Y^{+\widehat{N}_Y}\) would both just be the canonical lifting of \(\sigma_Y \upharpoonright \gamma_Y\)). But then

\(^{15}\)\(\sigma'_{XY}\) is defined by \([\xi, f]_{\sigma_{XY}} \mapsto \tilde{\sigma}(f)(\sigma_Y(\xi))\), where \(f\) is a good \(\Sigma_1^{(n-1)}(\widehat{N}_X)\) function with \(\text{dom}(f) < \gamma_X\) and \(\xi < \sigma_{XY}(\text{dom}(f))\).
\[ \gamma^+ \mathcal{N}_Y = \sup(\tilde{\sigma}_Y[\gamma^+_Y]) = \sup(\sigma'_{XY}[\gamma^+_Y]) < \sigma'_{XY}(\gamma^+_Y) = \gamma^+_X, \]
which is a contradiction.

**Claim 55.** There are sets \( T \) and \( \langle E_X | X \in T \rangle \) such that:

1. \( T \) is a stationary subset of \( S \)
2. For every \( X \in T \):
   a. \( E_X \subseteq D_X \), \( \text{otp}(E_X) = cf(\gamma) \), and \( E_X \) is cofinal in \( \gamma_X \).
   b. If \( cf(\gamma) > \omega \) then every element of \( E_X \) has cofinality \( < cf(\gamma) \) (recall that each \( X \) is closed under increasing sequences of length \( < cf(\gamma) \); so every element of \( E_X \) is mapped continuously by \( \sigma_X \)).
3. For every pair \( X \subset Y \) in \( T \), \( E_Y \subset \text{range}(\sigma_{XY}) \)

**Proof.** If \( cf(\gamma) = \omega \): for each \( X \in S \), pick some cofinal \( E_X \subseteq D_X \) of ordertype \( \omega \), and let \( E_X = \sigma_X[E_X] \). Since \( X \) is weakly internally approachable and \( |E'_X| = \omega \), there is some \( Z_X \) which is both an element and subset of \( X \) such that \( E'_X \subseteq Z_X \). Use the Fodor Lemma for \( P_n(H_\chi) \) to obtain a stationary \( T \subseteq S \) such that \( Z_X \) is the same for all \( X \in T \).

If \( cf(\gamma) > \omega \): fix some set \( B \) which is club in \( \gamma \), \( \text{otp}(B) = cf(\gamma) \), and every element of \( B \) has cofinality \( < cf(\gamma) \). Let \( T = \{ X \in S | B \subset X \} \); i.e. those \( X \) such that \( B \subset \text{range}(\sigma_X) \). Recall that \( \sigma_X \) is continuous on cofinalities \( < cf(\gamma) \), which implies that \( \sigma_X^{-1}[B] \) is club in \( \gamma_X \). For \( X \in T \) let \( E_X = \sigma_X^{-1}[B] \cap D_X \), and \( E'_X = \sigma_X[E_X] \).

It is straightforward to check that the sets constructed satisfy the conclusions of the claim. \( \square \)

For \( X \in T \), let \( E'_X := \sigma_X[E_X] \).

**Claim 56.** For every pair \( X \subset Y \) in \( T \), \( E'_Y \) is eventually contained in \( D'_X \).

**Proof.** We already know by Claim (55) that \( E'_Y \subset \text{range}(\sigma_X) \); so to prove the current claim it suffices to show that \( \sigma_X^{-1}[E_Y] \) is eventually contained in \( D_X \). Now let \( \beta \) be any element of \( E_Y \) such that \( \beta > \sigma_X(\min(D_X)) \) and \( \beta > \eta_{XY} \), where \( \eta_{XY} < \gamma_Y \) is the parameter from which \( \tilde{h}^{n+1}_{XY} \) is defined over \( N^Y \) (i.e. \( \tilde{h}^{n+1}_{XY} = \tilde{h}^{n+1}_{XY}(\eta_{XY}) \); if \( N_{XY} = N^Y \) then just let \( \eta_{XY} = 0 \)); note this makes sense because \( N_{XY} \) is an initial segment of \( N_Y \). Then \( \tilde{\beta} := \sigma_Y^{-1}(\beta) \in D_X \). If not, then by Lemma 24 there is some \( \tilde{\xi} < \tilde{\beta} \) such that \( \tilde{h}^{n+1}_{XY}(\tilde{\xi}) \in [\tilde{\beta}, \gamma_X] \). Then since \( \tilde{\sigma}_{XY} \) is \( \Sigma^0_1 \) preserving, \( \tilde{h}^{n+1}_{XY}(\xi) = \tilde{h}^{n+1}_{XY}(\eta_{XY})(\xi) \in [\beta, \gamma_Y] \) where \( \xi := \sigma_{XY}(\tilde{\xi}) \); so \( \xi, \eta_{XY} \succ \beta \) and therefore \( \tilde{h}^{n+1}_{XY}[\beta] \cap [\beta, \gamma_Y] \neq \emptyset \). This contradicts Lemma 24 and the fact that \( \beta \in D_Y \). \( \square \)

**Claim 57.** \( U'_X \subseteq U'_Y \) for every pair \( X \subset Y \) in \( T \).
Proof. CASE 1: There are cofinally many $\kappa \in E'_Y$ which are not measurable in $K$.

Let $F'_Y$ be a cofinal subset of such points in $E'_Y$. By the last claim, we can also assume that for every $\lambda \in F'_Y$, $\lambda_X := \sigma_X^{-1}(\lambda) \in D_X$. By elementarity of $\sigma_X$, for every $\lambda \in F'_Y$, $\lambda_X$ is not measurable in $K_X$ and $\sigma_Y^{-1}(\lambda)$ is not measurable in $K_Y$. This means that at the stage in $K$ vs. $K_X$ where $\lambda_X$ was the critical point, the extender applied on the $K$ side was the one of order zero (since $K_X$ is not moved in the coiteration), and similarly for the $K$ vs. $K_Y$ coiteration. So:

1. $\sigma_X^{-1}[F'_Y]$ is a generating sequence for $U_X$; i.e. $z \in U_X$ iff $\sigma_X^{-1}[F'_Y]$ is eventually contained in $z$.
2. $\sigma_Y^{-1}[F'_Y]$ is a generating sequence for $U_Y$; i.e. $z \in U_Y$ iff $\sigma_Y^{-1}[F'_Y]$ is eventually contained in $z$.

Item 1 implies that $F'_Y$ is a generating sequence for $U'_X$, and item 2 implies that it is a generating sequence for $U'_Y$, so $U'_X \subseteq U'_Y$. To see that $F'_Y$ is a generating sequence for $U'_X$ (the proof that it is a generating sequence for $U'_Y$ is similar): notice that $\tilde{\sigma}_X$ is $n$-cofinal and $\tau_X \leq \omega \rho^n_{N_X}$, so $\tilde{\sigma}_X$ is continuous at $\tau_X$. Then for any $z \in U'_X$, there is some $\delta < \tau_X$ such that $z \in \tilde{\sigma}(J^E_{\delta}^{N_X})$. Since $U_X$ is weakly amenable with respect to $N_X$ (this is one of the requirements to be on a premouse extender sequence), $U_X \cap J^E_{\delta}^{N_X}$ is an element of $\overline{N_X}$; let $F : \gamma_X \to_{onto} U_X \cap J^E_{\delta}^{N_X}$ be an enumeration in $N_X$. Then $\Delta F \in U_X$, so it contains a tail end of $\sigma_X^{-1}[F'_Y]$. So $\tilde{\sigma}(\Delta F) = \Delta(\tilde{\sigma}(F))$ contains a tail end of $F'_Y$ (recall $F'_Y \subseteq E'_Y \subset range(\sigma_X)$), say above $\lambda_0$. Also, since $z \in U'_X \cap \tilde{\sigma}(J^E_{\delta}^{N_X})$ then it is on the $\tilde{\sigma}(F)$ enumeration, so $\Delta \tilde{\sigma}(F)$ is eventually contained in $z$; say this happens at all points above $\lambda_1$. Then $z$ contains all points in $F'_Y$ above $\max(\lambda_0, \lambda_1)$.

CASE 2: Cofinally many $\lambda \in E'_Y$ are measurable in $K$. The proof is similar to case 1. Let $F'_Y$ be a cofinal sequence of such points in $E'_Y$, and again we can assume by the last claim that for every $\lambda \in F'_Y$, $\lambda_X := \sigma_X^{-1}(\lambda)$ is a critical point in the $K$ vs. $K_X$ coiteration. For every $\lambda \in F'_Y$, $\lambda_X$ is measurable in $K_X$ and $\lambda_Y := \sigma_Y^{-1}(\lambda)$ is measurable in $K_Y$. So at the stage in $K$ vs. $K_X$ where $\lambda_X$ was the critical point, the extender applied on the $K$ side had order $> 0$ (since the $K_X$ side did not move), and similarly for the $K$ vs. $K_Y$ coiteration. In particular, at such stages the $K$ side and the $K_X$ side have the same extender of order zero on the critical point. So:

1. $\sigma_X^{-1}[F'_Y]$ generates $U_X$ in the following sense: $z \in U_X$ iff for all sufficiently large $\lambda \in F'_Y$, $z \cap \lambda \in \mathcal{U}_{\mathcal{N}_X}^{\mathcal{U}_K}(\lambda, 0) = \mathcal{U}_K^X(\lambda, 0)$. 

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(2) $\sigma^{-1}_Y[F_Y']$ generates $U_Y$ in the following sense: $z \in U_Y$ iff for all sufficiently large $\lambda \in F_Y'$, $z \cap \lambda \in U^{N_Y}(\lambda, 0) = U^{K_Y}(\lambda, 0)$.

Item (1) implies that $F_Y'$ generates $U_X'$ in the analogous way; i.e. $z \in U_X'$ iff for sufficiently large $\lambda \in F_Y'$, $z \cap \lambda \in U^{N_X}(\lambda, 0) = U^{K}(\lambda, 0)$. $F_Y'$ generates $U_Y'$ in the analogous way. The proof of these is similar to case 1. So then $U_X' \subseteq U_Y'$. □

Recall that (the extender generated by) $U_X'$ is on $N_X'$’s extender sequence, and so is normal and weakly amenable with respect to $N_X'$. Also, $N_X'$ is an initial segment of $K$. Using these facts with Claim (57), it is straightforward to see that $U := \bigcup_{X \in T} U_X$ is an ultrafilter on $P(K(\gamma))$ which is normal and weakly amenable with respect to $K$ and concentrates on non-measurables. By Lemma 42 (the Weak Covering Lemma) $\gamma^+K$ has uncountable cofinality. So by Lemma 30 and Corollary 29, $\text{ult}(K,U)$ is wellfounded and $U$ generates $K$’s extender of order zero.

5.2. Proof of Theorem 46

5.2.1. Proof of part 1 of Theorem 46. Suppose for a contradiction that the first conclusion of Theorem 46 fails, and let $\beta < cf(\gamma)$ be the least ordinal where it fails. Then $S_\beta := \{\lambda < \gamma | o^K_\gamma(\lambda) = \beta\}$ is stationary in $\gamma$. We will obtain a contradiction by constructing a $K$-correct measure of order $\beta$ on some element of $S_\beta$.

The proof is similar to the previous section; in particular, all lemmas and claims from that section hold here, since we are now proving a special case of Theorem 45. Let $S_X^\beta := \sigma^{-1}_X[S_\beta]$; then $S_X^\beta$ is stationary in $\gamma_X$ since $\sigma_X$ is continuous on cofinalities $< cf(\gamma)$ and $S_\beta$ is stationary in $\gamma$. We make the following additional requirement on the sets $\langle E_X | X \in T \rangle$ constructed in Claim 55:

\begin{equation}
E_X \subset D_X \cap S_X^\beta
\end{equation}

This is possible because $S_\beta$ is stationary in $\gamma$ (in particular, the set $B$ from the proof of Claim 55 can be taken as a subset of $S_\beta$ in the present case).

From now on assume every $X \in S$ has all the nice properties from the last section (e.g. $K_X$ is not moved in its coiteration with $K$, $D_X$ is club in $\gamma_X$, etc.). For those $X \in S$ with $\beta \in X$, let $S_X^\beta = \sigma^{-1}_X(S_\beta)$. Since $\sigma_X$ is continuous on cofinalities $< cf(\gamma)$, then $S_X^\beta$ is stationary in $\gamma_X$ from $V$’s perspective and so $S_X^\beta \cap D_X$ is stationary in $\gamma_X$. $S_X^\beta \cap D_X$ consists of critical points of the coiteration $K$ vs. $K_X$ which lie on a common thread, and whose order in $K_X$ is exactly $\beta$. Furthermore, $\beta < \gamma$ and $D_X$ is a thread of critical points to $\gamma_X$, so eventually the critical points...
of the coiteration are strictly above $\beta_X := \sigma_X^{-1}(\beta) (= \beta; \text{ though this fact isn't used})$. Thus if $\kappa_i^X, \kappa_j^X$ are two sufficiently large critical points in $S^X_\beta \cap D_X$ with $0 < i < j$, then $\pi_{ij}^X(\nu_i^X) = \nu_j^X$ since there is no truncation at stage $\bar{\Omega}$ because since there is no truncation at stage $\bar{\Omega}$.

Then we would argue that the $\nu$ of $\nu$ because since there is no truncation at stage $\bar{\Omega}$...

Thus if $\kappa$...Thus if $\kappa$...Thus if $\kappa$...Thus if $\kappa$

This implies:

$$\text{Whenever } \kappa_i^X \text{ is an element and limit of } D_X \cap S^X_\beta, \quad \text{(30)} \quad \text{the set } D_X \cap S^X_\beta \cap \kappa_i^X \text{ is a generating sequence for } U^N_X(\kappa_i^X, \beta).$$

For each $X \in T$, select some $\lambda_X$ which is an element and limit of the stationary set $E_X$; and say $\lambda_X$ is the critical point used at stage $\bar{\Omega}_X$ (i.e. $\lambda_X = \kappa^X_{\bar{\Omega}_X}$). Also, WLOG assume there are no truncations for stages $j \in [\bar{\Omega}_X, \Omega_X)$ (this is possible because $\Omega_X$ is a limit ordinal). Let $N_X := N^X_{\bar{\Omega}_X} \quad \;$

Note that $o^N_X(\lambda_X) > \beta_X = o^K_X(\lambda_X)$ and $D_X \cap S^X_\beta \cap \lambda_X$ is a generating sequence for $U^N_X(\lambda_X, \beta)$, by (30).

Apply Fodor’s Lemma to find a stationary $T' \subset T$ such that $\sigma_X(\lambda_X)$ is the same for every $X \in T'$; call this common object $\bar{\lambda}$. Now $\sigma_X$ maps $\lambda_X$ cofinally to $\bar{\lambda}$, since $\lambda_X \in E_X$ and $\sigma_X$ is continuous on cofinalities $< cf(\gamma)$. Also, $\lambda_X$ is measurable in $\bar{N}_X$ and so $o^N_X(\mu) < \lambda_X$ for every $\mu < \lambda_X$. So by Lemma 43, for almost every $X \in T'$, $N'_{\bar{X}} = \text{ult}^*(\bar{N}_X, \sigma_X \upharpoonright \lambda_X)$ is an initial segment of $K$. Let $\bar{U}_X$ denote $U^N_X(\lambda_X, \beta_X)$ and $\bar{U}'_X$ denote the image of $\bar{U}_X$ via $\bar{\sigma}_X$. By elementarity of $\bar{\sigma}_X$ and the fact that $\beta_X < \lambda_X$, $\bar{U}'_X$ has Mitchell order $\beta$ in $\bar{N}'_X$. Also, $E'_X$ is a generating sequence for it; the proof is similar to Case 1 of Claim 57.

Next we show that whenever $X \subset Y$ are both in $T'$, $\bar{U}_X \subset \bar{U}'_X$. Similarly to the proof of Theorem 45, using an interpolation-like argument we construct a map $\bar{\sigma}_{XY} : \bar{N}_X \rightarrow \bar{N}_Y$ which extends $\sigma_{XY} \upharpoonright \lambda_X$ (note this latter map is cofinal into $\lambda_Y$), and argue that $\bar{N}_XY$ is an

16Unlike the scenario in section 45, it is always the case that $N^X_{\bar{\Omega}_X} = (N^X_{\bar{\Omega}_X})^*$ since we chose $\bar{\Omega}_X$ to be a stage past all truncations. This allows a construction of the $K$-extender to be a bit more direct than what is presented in this paper. In particular, instead of lifting $\sigma_X \upharpoonright \lambda_X$ to $N^X_{\bar{\Omega}_X}$ as we do below, one could instead lift $\sigma_X \upharpoonright (\nu^X_{\bar{\Omega}_X})^{+K_X}$ to a map $\bar{\sigma}_X$ with domain $N^X_{\bar{\Omega}_X + 1}$; this is possible in part because since there is no truncation at stage $\bar{\Omega}_X + 1$, then $K_X$’s cardinal successor of $\nu^X_{\bar{\Omega}_X}$ is also cardinal in $N^X_{\bar{\Omega}_X + 1}$. Then for each $X \in T'$, one could define a map $F_X : P^K(\bar{\lambda}) \cap \text{range}(\sigma_X) \rightarrow P(o^K(\bar{\lambda}))$ by $z \mapsto \bar{\sigma}_X \circ \bar{F}_X \circ \sigma_X^{-1}(z)$, where $\bar{F}_X = E^N_{\nu^X_{\bar{\Omega}_X}}$. Then we would argue that the $F_X$ cohere with each other, and $\bigcup_{X \in T'} F_X$ would be a $\langle \bar{\lambda}, o^K(\bar{\lambda}) \rangle$-extender which is on $K$’s extender sequence, which is a contradiction.
initial segment of $N_Y$ (note $N_{XY}$ and $N_Y$ coiterate above $\lambda_Y$, as this ordinal is measurable in both and thus does not index an extender in either). Also, similarly to Claim 56 we have:

For every pair $X \subset Y$ in $T'$, $\sigma_{XY}^{-1}[E_Y \cap \lambda_Y]$ is a cofinal subset of $\lambda_X$ which is eventually contained in $D_X \cap \lambda_X$.

Since $E_Y \subset S_\beta^X \cap \text{range}(\sigma_{XY})$, (5.2.1) implies that:

\[ \sigma_{XY}^{-1}[E_Y \cap \lambda_Y] \text{ is eventually contained in } D_X \cap S_\beta^X \cap \lambda_X \]

Recall from (30) that $D_X \cap S_\beta^X \cap \lambda_X$ is a generating sequence for $\bar{U}_X$ (the measure on $\lambda_X$ of order $\beta_X$ in $N_X$), and also that $E_Y \cap \lambda_Y$ is a generating sequence for $\bar{U}_Y$ (because $E_Y \cap \lambda_Y$ is a cofinal subset of $D_Y \cap S_\beta^Y \cap \lambda_Y$). Recall also that the pointwise image of a generating sequence for $\bar{U}_X$ is a generating sequence for $\bar{U}'_X$ (and similarly for $U_Y$ of course). These facts, together with (31), imply that for every pair $X \subset Y$ in $T'$, there is a generating sequence of $\bar{U}'_X$—namely $E_{\bar{U}}$—which is eventually contained in the generating sequence $\sigma_X[D_X \cap S_\beta^X \cap \lambda_X]$ of $\bar{U}'_X$. So $\bar{U}'_X \subset \bar{U}'_Y$.

As in the proof of Theorem 45, $\bar{U} := \bigcup_{X \in T_1} \bar{U}'_X$ is a normal, weakly amenable $K$-ultrafilter over $\bar{\lambda}$. By Lemma 30, $\text{ult}(K,U)$ is wellfounded; let $j : K \rightarrow_U K'$. Since each $\bar{U}'_X$ generated $N'_Y$’s (coherent) extender of Mitchell order $\beta$, then it is easy to see that $\sigma^K_{\bar{\lambda}}(\bar{\lambda}) = \beta$. Then by Corollary 29, $\bar{U}$ generates $E_{\sigma^K_{\bar{\lambda}}(\bar{\lambda})}$ which has Mitchell order $\beta$; so $\sigma^K_{\bar{\lambda}}(\bar{\lambda}) > \beta$, a contradiction.

5.2.2. Proof of part 2 of Theorem 46. For part 2, we need to show that for every $\beta < cf^V(\gamma)$, $K$ has a normal measure on $\gamma$ which concentrates on $a_\beta := \{ \xi < \gamma | \sigma^K_\xi(\xi) = \beta \}$. Let $S_{a_\beta}$ be the collection of $\lambda < \gamma$ such that $K$ has a normal measure on $\lambda$ concentrating on $a_\beta \cap \lambda$; by part 1 of Theorem 46, $S_{a_\beta}$ contains a club (in $V$). By Theorem 47 (proved below), $K$ has a normal measure on $\gamma$ which concentrates on $a_\beta$; i.e. the Mitchell order of this measure is $\beta$, which finishes the proof of Theorem 46 modulo the proof of Theorem 47.

5.3. Proof of Theorem 47. Let $a$ and $S_a$ be as in the hypothesis of Theorem 47, i.e. $a \in P^K(\gamma)$, $S_a$ is a $V$-stationary subset of $\gamma$, and for every $\lambda \in S_a$ there is a normal $K$-measure concentrating on $a \cap \lambda$. For each $\lambda \in S_a$, pick a normal measure $\mathcal{W}_\lambda \in K$ which concentrates on $a \cap \lambda$. By Corollary 29 $\mathcal{W}_\lambda$ generates an extender on $K$’s extenders sequence, say the extender indexed by $\mu_\lambda$; i.e. $\mathcal{W}_\lambda = E^K_{\mu_a, \lambda}$. For simplicity, choose $\mathcal{W}_\lambda$ to be the minimal measure concentrating on $a \cap \lambda$. 

As in Lemma \[51\], for every \(X \in S\) there is a closed unbounded set \(D_X \subset \gamma_X\) consisting of critical points of the coiteration which all lie on a common thread to \(\gamma_X\). Use the same notation as section \[5.1\], in particular, \(N_X := N_{\Omega X}\). Let \(S^X_a := \sigma^{-1}_X[S]\); since \(\sigma_X\) is continuous on \(\text{cof}(\omega)\) then \(S^X_a\) is stationary in \(\gamma_X\) from \(V\)’s perspective. If \(\kappa^X_i \in D_X \cap S^X_a\), then elementarity of \(\sigma_X\) implies that \(K_X\) has a measure on \(\kappa^X_i\) which concentrates on \(a_X := \sigma^{-1}_X(a)\), and the least such measure (in \(K_X\)’s Mitchell order) generates the extender indexed by \(\mu^X_i := \sigma^{-1}_X(\mu_{\sigma_X(\kappa^X_i)})\). In particular, \(\mu^X_i < o^{K_X}(\kappa^X_i) = \nu^X_i\) (recall the last equality is because the \(K_X\) side does not move), so:

\begin{equation}
\text{(32)} \quad \text{For every } \kappa^X_i \in S^X_a \cap D_X: \quad E_{\mu^X_i}^{K_X} = E_{\nu^X_i}^{N_X} = E^{N_X}_{\nu^X_i}.
\end{equation}

Let \(\mathcal{W}^X_i\) denote the measure from (32). Then (32) and the elementarity of the iteration maps implies that:

There is a normal measure \(\mathcal{W}_X\) such that:

- \(\text{cr}(\mathcal{W}_X) = \gamma_X\) and \(\mathcal{W}_X\) generates an extender on \(N_X\)’s extender sequence;
- \(a_X \in \mathcal{W}_X\);
- For every \(z \in P^{N_X}(\gamma_X)\) there is a pair \(z \in \mathcal{W}_X\), iff \(z \cap \kappa^X_i \in \mathcal{W}^X_i\) for sufficiently large \(\kappa^X_i\) in the stationary set \(D_X \cap S^X_a\).

Define a filter \(\mathcal{W}\) on \(P^K(\gamma)\) as follows:

\begin{equation}
\text{(34)} \quad z \in \mathcal{W} \text{ iff } \sigma^{-1}_X(z) \in \mathcal{W}_X \text{ for some } X \in S.
\end{equation}

We prove that \(\mathcal{W}\) is a \(K\)-ultrafilter, it is \(K\)-correct, and that its Mitchell order is exactly \(\beta\). Clearly, \(a \in \mathcal{W}\) since \(a_X \in \mathcal{W}_X\) for every \(X \in S\).

**Claim 58.** \(\mathcal{W}\) is a \(K\)-ultrafilter.

**Proof.** It is clear from the definition of \(\mathcal{W}\) that for every \(z \in P^K(\gamma)\) at least one of \(z, \gamma - z\) is in \(\mathcal{W}\). It remains to show that \(\mathcal{W}\) is proper. Suppose to the contrary that for some \(z \in P^K(\gamma)\) there is a pair \(X, X' \in S\) such that \(z_X \in \mathcal{W}_X\) but \((\gamma_X - z_{X'}) \in \mathcal{W}_{X'}\), where \(z_X := \sigma^{-1}_X(z)\) and \(z_{X'} := \sigma^{-1}_{X'}(z)\). By (33), this implies:

1. \(z_X \cap \kappa^X_i \in \mathcal{W}^X_i\) for sufficiently large \(\kappa^X_i \in D_X \cap S^X_a\).
2. \((\kappa^X_j - z_{X'}) \in \mathcal{W}^{X'}_{j}\) for sufficiently large \(\kappa^X_j \in D_{X'} \cap S^X_{a'}\).

Now select a sufficiently large element \(\kappa\) of the \(V\)-stationary set \(\sigma_X[D_X] \cap S_a \cap \sigma_X[D_{X'}]\). Item (1) and elementarity of \(\sigma_X\) imply that \(z \cap \kappa \in E^K_{\mu_{\kappa,\kappa}}\). But item (2) and elementarity of \(\sigma_{X'}\) imply that \((\kappa - z) \in E^K_{\mu_{\kappa,\kappa}}\). This contradicts that \(E^K_{\mu_{\kappa,\kappa}}\) is an ultrafilter. \(\square\)
Claim 59. \( W \) is normal with respect to \( K \); i.e. whenever \( f \in K \), \( \text{dom}(f) = \gamma \) and \( \text{rng}(f) \subset W \) then \( \Delta f \in W \).

Proof. Let \( \vec{A} \in K \) be a \( \gamma \)-sequence of sets in \( W \). Pick any \( X \in S \) with \( \vec{A} \in X \). Then \( \vec{A}_X := \sigma_X^{-1}(\vec{A}) \in K_X \) is a \( \gamma_X \)-sequence of subsets of \( \gamma_X \), and \( \vec{A}_X \in N_X \) since \( P^{K_X}(\gamma_X) \subseteq P^{N_X}(\gamma_X) \) (the latter is because \( K_X \) does not move in the coiteration and the least disagreement (if there is one) between \( N_X \) and \( K_X \) has critical point \( \geq \gamma_X \)). By the definition of \( W \) and the fact that every set in \( \vec{A} \) is in \( W \), then every set on the sequence \( \vec{A}_X \) is in the measure \( W_X \). Since \( W_X \) is normal with respect to \( N_X \) (recall it generates an extender on \( N_X \)’s extender sequence; see \( (33) \)), \( \Delta \vec{A}_X = \sigma_X^{-1}(\Delta \vec{A}) \in W_X \). So \( \Delta \vec{A} \in W \) by the definition of \( W \). \( \square \)

Claim 60. \( \text{ult}(K, W) \) is wellfounded

Proof. We already know by Theorem [45] that \( \gamma \) is measurable in \( K \) and so \( \gamma \) does not index an extender in \( K \). And by Lemma [42] (Weak Covering Lemma) \( \tau := \gamma^+K \) has uncountable cofinality. So by Corollary [19] it suffices to show that \( \text{ult}(K|\tau, W) \) is wellfounded.

Suppose to the contrary, and let \( \{f_n|n \in \omega\} \subseteq K|\tau \) witness the illfoundedness of \( \text{ult}(K|\tau, W) \). Since \( \gamma \) is the largest cardinal in \( K|\tau \), then WLOG we can assume \( \text{range}(f_n) \subseteq \gamma \) for every \( n \in \omega \). Pick an \( X \in S \) with \( \{f_n|n \in \omega\} \subseteq X \). Let \( \tau_X := \sigma_X^{-1}(\tau) \). Then \( \{\sigma_X^{-1}(f_n)|n \in \omega\} \) witnesses that \( \text{ult}(K_X|\tau_X, W_X) \) is illfounded; but this is a contradiction, since \( K_X|\tau_X \) is an initial segment of the mouse \( N_X \) and \( W_X \) generates an extender on \( N_X \)’s extender sequence. \( \square \)

Finally, we note that by lifting stationary sets the measure \( W \) can be equivalently defined as follows:

\[ z \in W \text{ iff } z \cap \lambda \in E^K_{\mu, \kappa} \text{ for all but } V \text{-nonstationarily many } \kappa < \gamma \]
6. AN APPLICATION TO STATIONARY SET REFLECTION

6.1. Reflection of a single stationary set at a small cofinality.

If $\kappa$ is an ordinal of uncountable cofinality, $S \subseteq \kappa$ is stationary, and $\gamma < \kappa$ has uncountable cofinality, we say $S$ reflects at $\gamma$ if $S \cap \gamma$ is stationary in $\gamma$. Starting from a Mahlo cardinal, Harrington and Shelah in [4] obtained a model of “Every stationary subset of $\omega_2 \cap \text{cof}(\omega)$ reflects.” In the other direction, if every stationary subset of $\omega_2 \cap \text{cof}(\omega)$ reflects, then Jensen’s global square sequence for $L$ can be used to show that $\omega_2$ is Mahlo in $L$. Reflection for stationary subsets of a successor cardinal $\geq \omega_3$ are similarly equiconsistent with a Mahlo cardinal. However, requiring the reflection points to have small cofinality yields larger cardinals in $K$:

**Theorem 61.** Assume every stationary subset of $S_0^3$ reflects at a point of cofinality $\omega_1$. For $f, g \in \omega \times V$, define $f <^* g$ iff $f(\xi) < g(\xi)$ for almost every $\xi \in \omega_3 \cap \text{cof}(< \omega_2)$. Then $h_\nu <^* o^K_\nu$ for each $\nu < \omega_3^+ K$. 

First, the hypothesis of Theorem (61) is equivalent to requiring that $S$ reflect stationarily often in $S_0^3$: Suppose $S \subset S_0^3$ is stationary and reflects only nonstationarily often in $\text{cof}(\omega_1)$. Let $D \subset \omega_3$ be an unbounded subset of $\omega_3$ which is closed under $\omega_1$-limits, and such that for every $\gamma \in D$, $S \cap \gamma$ is nonstationary. Then $S \cap \text{Lim}(D)$ is a stationary subset of $S_0^3$ which reflects nowhere in $\text{cof}(\omega_1)$. So:

(36) Every stationary subset of $S_0^3$ reflects stationarily often in $\text{cof}(\omega_1)$.

**Claim 62.** Almost every $\gamma \in S_1^3$ is regular in $K$.

*Proof.* We will use $K$’s “global square” sequence. For the $K$ in the current paper (i.e. for one strong cardinal) the construction in Zeman [11] suffices; see Schimmerling and Zeman [8] and Zeman [12] for much stronger versions. The global square sequence is a sequence $\langle c_\alpha | \alpha \in \text{Sing}(K) \rangle$, where $\text{Sing}(K)$ is the collection of all ordinals which are singular in $K$. This sequence has the following properties:

1. For every $\alpha \in \text{Sing}(K)$: $c_\alpha \subset \text{Sing}(K)$, $\text{otp}(c_\alpha) < \alpha$, and $c_\alpha$ is closed and unbounded in $\alpha$.

2. If $\alpha$ is singular in $K$ and $\beta$ is a limit of $c_\alpha$, then $c_\beta = c_\alpha \cap \beta$.

18Recall from Definition [6] that for mice below 0-pistol, $o^M(\kappa)$ denotes the “Mitchell order of $\kappa$ in $M$”; more precisely, the ordertype of the collection of $\nu \geq \kappa^+ M$ which index an extender with critical point $\kappa$. On the other hand $o^M(\kappa)$ denotes the least primitively recursively closed ordinal which does not index an extender with critical point $\kappa$. 

Suppose for a contradiction that $T := \{ \gamma \in S_1^3 | \gamma \in \text{Sing}(K) \}$ is stationary. So for every $\gamma \in T$, $c_\gamma$ is defined and $c_\gamma \cap \text{cof}^V(\omega)$ is a club in $\gamma$. Let $S := \bigcup_{\gamma \in T} \text{lim}(c_\gamma \cap \text{cof}^V(\omega))$; this is a stationary subset of $S_0^3$, and for every $\alpha \in S$, $\alpha$ is singular in $K$ and so $c_\alpha$ is defined and has ordertype $< \alpha$. By the Fodor lemma there is a $\beta < \omega_3$ and a stationary $\bar{S} \subset S$ such that $\text{otp}(c_\alpha) = \beta$ for every $\alpha \in \bar{S}$.

Then $\bar{S}$ has no reflection point in $S_1^3$. Suppose $\gamma \in S_1^3$ were a reflection point.

**CASE 1:** $\gamma$ is regular in $K$. Then by Theorem 46, there is (in $V$) a club $d \subset \gamma$ consisting of $K$-measurables; but then $d$ cannot intersect $\bar{S}$, since every element of $\bar{S}$ is singular in $K$.

**CASE 2:** $\gamma$ is singular in $K$. Then $c_\gamma$ is defined. Then $\bar{S} \cap \text{lim}(c_\gamma)$ has at most one element: suppose $\alpha < \alpha'$ were both in $\bar{S} \cap \text{lim}(c_\gamma)$. Since $\alpha'$ is a limit of $c_\alpha$, then $c_{\alpha'} = c_\gamma \cap \alpha'$. Then, since $\alpha$ is a limit of $c_\alpha$ below $\alpha'$, it is also a limit of $c_{\alpha'}$ and so $c_\alpha = c_{\alpha'} \cap \alpha$. So $\text{otp}(c_\alpha) < \text{otp}(c_{\alpha'})$, contradicting that $\alpha$ and $\alpha'$ are both in $\bar{S}$. $\square$(Claim 62)

Let $R_1^3 := S_1^3 \cap \text{Reg}(K)$; by claim 62 this is almost all of $S_1^3$. We will first prove a restricted version of Theorem 61 for $\lambda \in \text{cof}(\omega)$:

**Claim 63.** Let $\nu < \omega_3^{+K}$ and $b_\nu \in P^K(\omega_3)$ code $\nu$. Then for almost every $\lambda \in S_3^3$: $\text{o}^K(\lambda) > \text{otp}(b_\nu \cap \lambda)$.

**Proof.** First, it is easy to show that the map $\lambda \mapsto \text{otp}(b_\nu \cap \lambda)$ is “the” $\nu$-th canonical function on $\omega_3$ (see section 2). We prove Claim 63 by induction on $\nu < \omega_3^{+W}$. For illustrative purposes, consider the base case $\nu = 0$: suppose $\{ \lambda \in S_0^3 | \text{o}^K(\lambda) > 0 \}$ did not contain an $\omega$-club, i.e. there is a stationary set $T$ of $\lambda \in S_0^3$ which are not measurable in $K$. By (36) and Claim 62 $T$ reflects at some $\gamma \in R_1^3$. But by Theorem 46, there is a club of $K$-measurables in $\gamma$, so this club cannot intersect $T$. Contradiction.

The basic outline of the base case is used for the inductive case. Suppose $\nu > 0$ and the claim holds for all $\tau < \nu$. Let $C_\tau \in V$ be an $\omega$ club in $\omega_3$ witnessing the claim for each $\tau < \nu$ (so $\text{o}^K(\lambda) > \text{otp}(b_\nu \cap \lambda)$ for every $\lambda \in C_\tau$), and fix some sequence $\langle \tau_i | i < \text{cf}(\nu) \rangle \in V$ cofinal in $\nu$; if $\nu$ is a successor ordinal we will define its cofinality as 1, and the sequence $\langle \tau_i | i < \text{cf}(\nu) \rangle$ will just be $\langle C_{\nu-1} \rangle$. Let $b_\nu \in P^K(\omega_3)$ code $\nu$, and $b_{\tau_i} \in P^K(\omega_3)$ code $\tau_i$. Fix a large regular $\theta > 2^{\omega_1}$ and consider the collection $\bar{C}$ of all $Z \in P_{\omega_3}(H_\theta)$ such that $Z \prec (H_\theta, \langle (\tau_i, C_{\tau_i}, b_{\tau_i}) | i < \text{cf}(\nu) \rangle, \ldots)$. For $Z \in \bar{C}$ let $\pi_Z : H_Z \rightarrow H_\theta$ be the inverse of the Mostowski collapse of $Z$. Then $\text{cr}(\pi_Z) = Z \cap \omega_3 = \lambda_Z$ and $\pi_Z^{-1}(\langle b_{\tau_i} | i < \text{cf}(\nu) \rangle) = \langle b_{\tau_i} \cap \lambda_Z | i < \lambda_Z \rangle$ (if $\text{cf}(\nu) = \omega_3$; similarly if
cf(ν) < ω₃). It follows that for all λ ∈ C := \{Z \cap ω₃|Z ∈ \hat{C}\}:

\begin{align*}
otp(b_ν \cap λ) &= \text{sup}_{i<λ}(otp(b_{τ_i} \cap λ)) & \text{if } cf(ν) = ω₃ \\
otp(b_ν \cap λ) &= \text{sup}_{i<cf(ν)}(otp(b_{τ_i} \cap λ)) & \text{if } cf(ν) \leq ω₂ \\
otp(b_ν \cap λ) &= otp(b_{ν₁} \cap λ) + 1 & \text{if } ν \text{ successor}
\end{align*}

(37)

Furthermore, since Z ≺ (H_θ, (C_{τ_i}|i < cf(ν)), ... ) for Z ∈ \hat{C} then λ_Z is a limit of C_{τ_i} for each i < λ_Z, so:

\begin{align*}
C \cap \text{cof}(ω) &\subset \bigg\{ \begin{array}{ll}
\Delta_{i<cf(ν)}C_{τ_i} & \text{if } cf(ν) = ω₃ \\
\cap_{i<cf(ν)}C_{τ_i} & \text{if } cf(ν) < ω₃ \\
C_{ν₁} & \text{if } ν \text{ successor}
\end{array} \bigg\}
\end{align*}

(38)

Then (37) and (38) imply that o^K(λ) ≥ otp(b_ν \cap λ) for every λ ∈ C \cap \text{cof}(ω). Now suppose for a contradiction that o^K(λ) ≤ otp(b_ν \cap λ) for stationarily many λ ∈ S_0^3; let T denote this stationary set. Let S_ν := T ∩ C. Then:

\begin{align*}
o^K(λ) &= otp(b_ν \cap λ) & \text{for every } λ ∈ S_ν.
\end{align*}

(39)

We will obtain a contradiction by finding some λ ∈ S_ν whose K-order is > otp(b_ν \cap λ).

By (36) and Claim 62

\begin{align*}
\text{Ref} := \{ \gamma ∈ R^3_1|S_ν \text{ reflects at } γ \} \text{ is stationary.}
\end{align*}

(40)

Let S be a stationary set of weakly internally approachable sets in P_{ω₂}(H_χ) such that γ'_X := sup(X ∩ ω₃) ∈ Ref and X ≺ (H_χ, ∈, ...); this is possible since Ref is a stationary subset of S_0^3. By Lemma 39 we can WLOG assume that for every X ∈ S the K_X side does not move in its coiteration with K, and the K side truncates by stage 1.

The remainder of the proof is similar to the proof of Theorem 45. The statements of all lemmas and claims through Lemma 24 can be repeated verbatim, and the proofs involve only a minor amendment: in the current proof, for each X ∈ S the set \{Y ∈ S|γ'_Y = γ'_X\} might be nonstationary, whereas in the proof of Theorem 45 all X ∈ S were cofinal in some fixed γ. This difference corresponds to how the parameter γ in the statement of Lemma 31 is interpreted in the proofs: in section 5 we interpreted γ as the ordinal in the hypothesis of Theorem 45 whereas in the current proof we interpret γ as ω₃.

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19 The reason for setting things up this way—rather than fixing a single reflection point γ—will become apparent in the proof in section 6.2. In the current section, we could instead work with a single reflection point γ of S_ν.

20 In fact, we could simply use Lemma 44 to argue that, e.g., Lemma 50 holds for every X ∈ S.
So for every $X \in S$:

- The critical points of the coiteration of $K$ with $K_X$ are cofinal in $\gamma_X := \sigma_X^{-1}(\omega_3)$ (= the cofinal preimage of $\gamma'_X$)

$$\tag{41}$$

- There is a club $D_X$ in $\gamma_X$ consisting of critical points of the coiteration of $K$ vs. $K_X$, which all lie on a common thread to $\gamma_X$.

Since each $\sigma_X$ is continuous on $\text{cof}(\omega)$ and $\gamma'_X$ is a reflection point of $S_\nu$, then $S_\nu^X := \sigma_X^{-1}[S_\nu \cap \gamma_X]$ is stationary in $\gamma_X$ from $V$’s perspective. Let PRC denote the class of primitive recursively closed ordinals. Now for every sufficiently large $\pi^X \in PRC$,

The Mitchell order of $E^{N^X_\nu} = \text{otp}(PRC \cap [\tau^X_i, \nu^X_i])$

$$\tag{42}$$

$$\text{otp}(PRC \cap [\tau^X_i, \sigma^X_i(\kappa^X_i)]) = \sigma^{-1}_X(\text{otp}(b_\nu \cap \sigma_X(\kappa^X_i))) = \text{otp}(\kappa^X_i \cap b_\nu)$$

where $b_\nu := \sigma^{-1}_X(b_\nu)$; also note that if $b_X \in \text{range}(\pi^X_{i,j,\Omega})$ then $\pi^X_{i,j,\Omega}(b_X \cap \kappa^X_i) = b_X \cap \gamma_X$ since $\kappa^X_i \in D_X$ (here $\Omega_X$ is defined as in the discussion around (24)). Now if $\kappa^X_i < \kappa^X_j$ are both in $D_X \cap S_\nu^X$ and $b_X \in \text{range}(\pi^X_{i,j,\Omega})$ then $\pi^X_{i,j}(b_X \cap \kappa^X_i) = b_X \cap \kappa^X_j$; so (42) implies that $\pi^X_{i,j}(\nu^X_i) = \nu^X_j$. In other words, the extenders applied at stages corresponding to $D_X \cap S_\nu^X$ all lie on a common thread. This implies:

For every $\kappa^X_i$ which is an element and limit of the stationary set $D_X \cap S_\nu^X : D_X \cap S_\nu^X \cap \kappa^X_i \in X$ is a generating sequence for $U^{N^X_\nu}(\kappa^X_i, \text{otp}(b_\nu \cap \kappa^X_i))$.

Pick some $\lambda_X$ which is both an element and limit of the stationary set $D_X \cap S_\nu^X$, and using Fodor’s Lemma on $P_{\omega_2}(H_\theta)$ we will WLOG assume that $\sigma_X(\lambda_X)$ is fixed at some $\bar{\lambda}$ for every $X \in S$. Let $\bar{X} \in S$ denote the stage of the coiteration where $\lambda_X$ is the critical point, and $\bar{N}_X := N^X_{\Omega_X}$; this notation closely follows that of section 5.2.1. Similarly to Lemma 55 there is a stationary $T \subset S$ and a collection of sets $\langle E_X | X \in T \rangle$ such that $E_X \subset D_X \cap S_\nu^X \cap \lambda_X$ and for every $X \subset Y$ in $T$, $E_Y \subset \text{range}(\sigma_{XY})$. Also, as in section 5.2.1 $\sigma_{XY} \upharpoonright \lambda_X$ can be lifted to a map $\tilde{\sigma}_{XY} : \bar{N}_X \rightarrow \bar{N}_{XY}$ for $X \subset Y$ in $T$ and:

$$\tag{44}$$

Recall the crucial part was showing $\sigma^{-1}_X[D_Y]$ is eventually contained in $D_X$; (44) then follows easily since clearly $\sigma^{-1}[S_Y^X] \subseteq S_\nu^X$.

\[\text{so that there are no truncations at stages in } [i, \Omega_X].\]
Let $\tilde{U}_X := \mathcal{U}^{\tilde{X}}(\lambda_X, otp(b_X \cap \lambda_X))$ and $\tilde{U}'_X$ be the image of $\tilde{U}_X$ under $\tilde{\sigma}_X$. Similarly to the discussion in section 5.2.1, the pointwise image of $D_X \cap S^{\nu}_Y \cap \lambda_X$ is a generating sequence for $\tilde{U}'_X$, and (44) guarantees that $\tilde{U}'_X \subseteq \tilde{U}'_Y$ whenever $X \subseteq Y$ are in $T$. Then $\tilde{U} := \bigcup_{X \in T} \tilde{U}'_X$ is a normal, weakly amenable $K$-ultrafilter on $\bar{\lambda}$, and by Lemma 30 $\text{ult}(K, \tilde{U})$ is wellfounded; let $j: K \rightarrow V' K'$. Now $\tilde{U}_X$ concentrates on $\{\xi < \lambda_X | o^{\tilde{X}}_\nu(\xi) = otp(b_X \cap \xi)\}$, so $\tilde{U}'_X$ concentrates on $\{\xi < \bar{\lambda} | o^{\tilde{X}}_\nu(\xi) = otp(b_v \cap \lambda)\} = \{\xi < \bar{\lambda} | o^K(\xi) = otp(b_v \cap \lambda)\}$ (since $N'_x$ is an initial segment of $K$). So $o^K(\bar{\lambda}) = otp(b_v \cap \bar{\lambda})$. By Corollary 29 $\tilde{U}$ generates the extender $E^K_{o^{K'}(\bar{\lambda})}$ which has Mitchell order $otp(b_v \cap \bar{\lambda})$, so $o^K(\bar{\lambda}) > otp(b_v \cap \bar{\lambda})$, contradicting the fact that $\bar{\lambda} \in S_v$. $\square$(Claim 63)

The rest of Theorem 61 follows from Claim 63 and Theorem 47. Let $\nu < \omega^+_3$ and $b \in P^K(\omega_3)$ code $\nu$. By Claim 63 there is an $\omega$-club $C \subseteq \omega_3$ such that $o^K(\lambda) > otp(b \cap \lambda)$ for every $\lambda \in C$; i.e. for every $\lambda \in C$ there is a $K$-measure on $\lambda$ which concentrates on $\lambda \cap \{\xi < \omega_3 | o^K(\xi) = otp(b \cap \xi)\}$. Then for any $\gamma \in S^3_1 \cap \text{Lim}(C)$, Theorem 47 guarantees that there is a $K$-measure on $\gamma$ which concentrates on $\{\xi < \gamma | o^K(\xi) = otp(b \cap \xi)\}$. This completes the proof of Theorem 61. 

6.2. Simultaneous reflection at a small cofinality. Simultaneous reflection of stationary sets is generally a stronger principle. Magidor [6] proved that the exact consistency strength of "every pair of stationary subsets of $\omega_3 \cap cof(\omega)$ have a common reflection point" is exactly a weakly compact cardinal (the forcing direction was a strengthening of Baumgartner [4]). Again, requiring the reflection points to have small cofinality results in a stronger statement:

**Theorem 64.** Assume every pair of stationary subsets of $S^3_0$ have a common reflection point in $S^3_1$. Then there is an inner model of $O(\kappa) = \kappa^+$. In particular, if $0$-pistol does not exist then $o^K(\omega_3) \geq \omega^+_{3}$. 

First, note that the hypothesis of the theorem is equivalent to:

(45) Every pair of stationary subsets of $S^3_0$ have stationarily many common reflection points in $cof(\omega_1)$.

To see (45), suppose $S, T \subseteq S^3_0$ are stationary, but have only nonstationarily many common reflection points in $cof(\omega_1)$. Let $D \subseteq S^3_1$ be unbounded and closed under limits of cofinality $\omega_1$, such that $S$ and $T$ have no common reflection points in $D$. Let $\bar{D}$ be the closure of $D$; i.e. $\bar{D} := D \cup \text{Lim}(D)$. Let $S' := S \cap \bar{D}$ and $T' = T \cap \bar{D}$. Then $S', T'$ have no common reflection points in $cof(\omega_1)$: let $\gamma \in cof(\omega_1)$ be a limit of both $S'$ and $T'$. Then $\gamma$ is an $\omega_1$-cofinal limit of $\bar{D}$, and thus $\gamma \in D$. 

So at least one of $S, T$ fails to reflect at $\gamma$. Since $S' \subseteq S$ and $T' \subseteq T$, then at least one of $S', T'$ fails to reflect at $\gamma$.

Fix some $b \in P^K(\omega_3)$ which codes a wellorder of $\omega_3$; the goal is to build a $K$-measure on $\omega_3$ of order $otp(b)$. Using Theorem \ref{61} let $C_b$ be an $\omega$-club in $\omega_3$ such that for every $\lambda \in C_b$, $o^K_*(\lambda) > otp(b \cap \lambda)$. Let $S$ be the collection of $X \in P_{\omega_2}(H_{\theta})$ such that $b \in X$, $X \cap \omega_2 \in S^b_1$, and $\gamma^X := sup(X \cap \omega_3) \in R^b_3$; note $S$ is stationary (in fact contains a subset which is closed under $\omega_1$-length chains). Let $\pi_X : H_X \rightarrow H_{\theta}, N_X, \gamma_X, etc. be as in the proof of Theorem \ref{61} and let $b_X := \pi_X^{-1}(b)$. For any element $\kappa^X_i$ of the club $G_X := D_X \cap \pi_X^{-1}(C_b)$: $\nu^X_i = o^K_*(\kappa^X_i) > otp(b_X \cap \kappa^X_i)$. This implies:

$$U^N_X(\gamma_X, otp(b_X)) \text{ exists, and } z \in U^N_X(\gamma_X, otp(b_X)) \text{ if and only if } z \cap \kappa \in U^K_X(\kappa, otp(b_X \cap \kappa)) \text{ for sufficiently large } \kappa \text{ in the club } G_X.$$  

Define the filter $F(\omega_3, b)$ on $P^K(\omega_3)$ by: $z \in F(\omega_3, b)$ if and only if $z_X \in U^N_X(\gamma_X, otp(b_X))$ for stationarily many $X \in S$ with $z \in X$, where $(z_X, b_X) = \pi_X^{-1}(z, b)$. This makes sense—i.e. $U^N_X(\gamma_X, otp(b_X))$ is defined—by \eqref{46}.

**Claim 65.** $F(\omega_3, b)$ is an ultrafilter on $P^K(\omega_3)$.

**Proof.** Since $S$ is stationary, $F(\omega_3, b)$ is clearly a filter; it remains to show that this filter is proper. Suppose not; so there is a $z \in P^K(\omega_3)$ such that both $S_z := \{X \in S \mid z \in X \text{ and } z_X \in U^N_X(\gamma_X, b_X)\}$ and $S^z := \{X \in S \mid z \in X \text{ and } \gamma_X - z_X \in U^N_X(\gamma_X, b_X)\}$ are stationary; also, $\{X \in S \mid z \in X\}$ is the disjoint union of $S_z$ and $S^z$. Let $\bar{S}_z := \bigcup_{X \in S_z} \sigma_X[G_X]$ and $\bar{S}^z := \bigcup_{X \in S^z} \sigma_X[G_X]$; these are stationary subsets of $S_0^3$. Note:

$$\text{(47)}$$

- For every $\lambda \in \bar{S}_z$: $z \cap \lambda \in U^K(\lambda, otp(b \cap \lambda))$.
- For every $\lambda \in \bar{S}^z$: $\lambda - z \in U^K(\lambda, otp(b \cap \lambda))$.

Let $\bar{R} := \{sup(X \cap \omega_3) \mid X \in S \text{ and } z \in X\}$; note $\bar{R}$ is almost all of $S^3$. By \eqref{45}, $\bar{S}_z$ and $\bar{S}^z$ have some common reflection point $\bar{\gamma} \in \bar{R}$. Fix a $X$ which witnesses that $\bar{\gamma} \in \bar{R}$; i.e. $X \in S$ and $z \in X$. Now $U^K(\bar{\gamma}, otp(b \cap \bar{\gamma}))$ exists so either $z \cap \bar{\gamma}$ or $\bar{\gamma} - z$ is in this measure; without loss of generality assume $z \cap \bar{\gamma}$ is in the club $G_X$. By Theorem \ref{47}, there is a club $d \cap \bar{\gamma}$ such that for all $\lambda \in d$, $z \cap \lambda \in U^K(\lambda, otp(b \cap \lambda))$. But $d$ cannot intersect $\bar{S}^z$, by \eqref{47}. This contradicts that $\bar{S}^z$ reflects at $\bar{\gamma}$. \hfill \square

**Claim 66.** $F(\omega_3, b)$ is normal with respect to $K$; i.e. if $\bar{A} \subseteq K$ is an $\omega_3$-sequence of sets from $F(\omega_3, b)$ then $\Delta \bar{A} \subseteq F(\omega_3, b)$.
Proof. The proof is the same as that of Claim 59.

Claim 67. $\text{ult}(K, F(\omega_3, b))$ is wellfounded.

Proof. Similar to proof of Claim 60.

Let $j : K \rightarrow_{F(\omega_3, b)} K'$; Claim 66 implies that $\omega_3 = \text{cr}(j)$. The definition of $F(\omega_3, b)$ guarantees that it concentrates on $\{\xi < \omega_3 | o^K(\xi) = \text{otp}(b \cap \xi)\}$; so $o^{K'}(\omega_3) = \text{otp}(b)$. By Corollary 29, $F(\omega_3, b)$ generates an extender on the $K$ sequence of Mitchell order $\text{otp}(b)$. This concludes the proof that $o^K(\omega_3) \geq \omega_3^+$. 
REFERENCES


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